BV and BFV formalism beyond perturbation theory

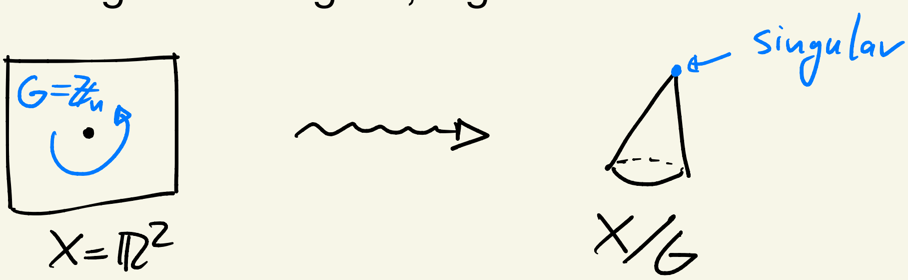
Based on joint works w/ Benini-Safronov [2104.14886] and Benini-Pridham [2201.10225]

What is derived geometry?

- Traditional geometric frameworks, such as manifolds or schemes, are incapable to describe certain important geometric objects:

(i) Quotients by non-free group actions:

X/G is in general singular, e.g.



and ignores in how many ways points get identified, e.g.

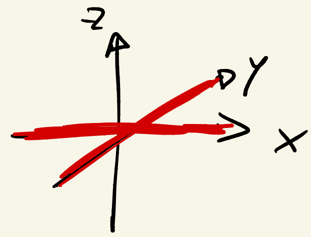
$$*/G = * \quad \text{independently of } G$$

(ii) Non-transversal intersections:

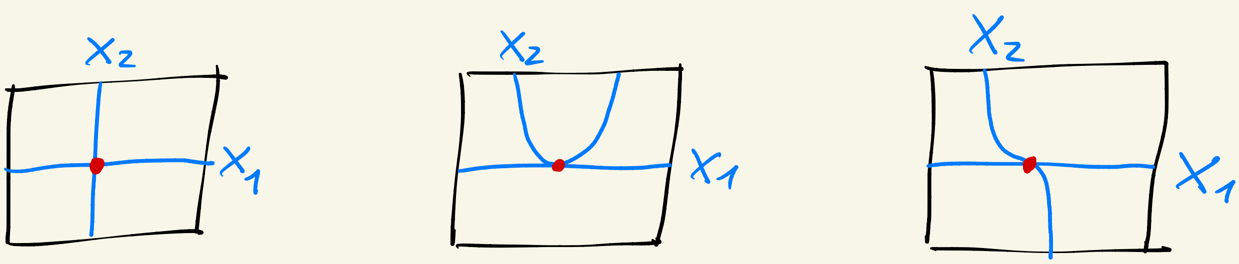
$X_1 \times_Y X_2$ is in general singular, e.g.

$$Y=\mathbb{R}^3\,,\quad X_1=\{(x,y,z)\in\mathbb{R}^3 : z=0\}\,,\quad X_2=\{(x,y,z)\in\mathbb{R}^3 : (xy)^2=z\}$$

$$\rightsquigarrow\; X_1 \times_Y X_2 = \{(x,y,z)\in\mathbb{R}^3 : z=0 \text{ and } (xy)^2=0\}$$



and ignores intersection multiplicities, e.g.



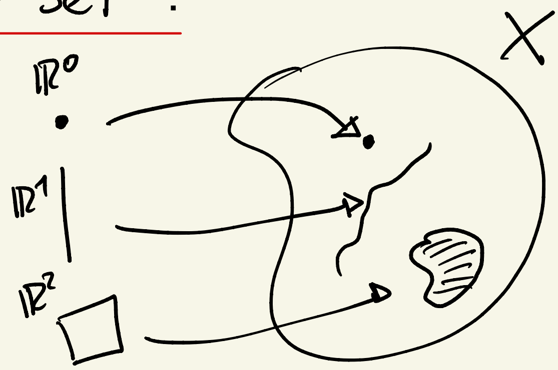
- Derived (algebraic) geometry resolves these issues by introducing a refined and powerful concept of space called derived stacks.

- To get some intuition, we have to recall functors of points:

$Aff := CAlg^{op}$		$Sh(Aff, Set)$
affine schemes (building blocks)	$\xhookrightarrow[\text{(fully faithful)}]{\text{Yoneda}}$	schemes and more (what you get from glueing)

Interpretation of a functor $X : Aff^{op} \longrightarrow Set$:

- $X(\mathbb{R}^0) =$ "points in X"
- $X(\mathbb{R}^1) =$ "curves in X"
- $X(\mathbb{R}^2) =$ "surfaces in X" ...



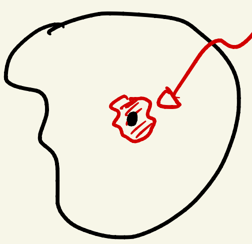
- Derived stacks have a richer functor of points:

improves intersections ⇝ EOM or constraints (antifields)

$$\begin{tikzcd}
Aff^{op} = CAlg \arrow[rr, "\text{scheme}"] \arrow[dd] \arrow[drr, "\text{stack}"] \arrow[ddrr, "\infty\text{-stack}"'] & & Set \arrow[d, hook] \\
& & Grpd \arrow[d, hook, dotted] \\
dAff^{op} := dgCAlg_{\geq 0} \arrow[rr, "\text{derived}", "(\infty\text{-})\text{stack}"'] & & \infty Grpd
\end{tikzcd}$$

improves quotients ⇝ gauge symmetry (ghosts)

- $\boxed{\text{BRST/BV/BFV/… = perturbative/formal aspects of derived geometry:}}$



formal neighborhood of point $\xleftrightarrow[\text{Lurie, Pridham}]{}$ L_∞-algebra (or DGC Alg)

$X \in dSt$

Application 1: Nonperturbative (classical and finite dim.) BV formalism

Physical scenario and motivation:

A physical system is typically described by

1.) a space of fields X

2.) an action of the gauge group $X \times G \longrightarrow X$

3.) a gauge-invariant action function $S : X \longrightarrow \mathbb{K}$

Want to determine the space of extrema of S modulo gauge symmetries, which involves taking quotients X/G and intersections $d^{dR}S = 0$.

$\rightsquigarrow$ $\boxed{\text{DAG potentially important for this problem!}}$

Mathematical formalization:

Consider a smooth affine scheme $X = \mathrm{Spec}\, A$ with an action $X \times G \longrightarrow X$ of a smooth affine group scheme $G = \mathrm{Spec}\, H$.

A gauge-invariant function $S : X \longrightarrow \mathbb{K}$ is the same datum as a function $S : [X/G] \longrightarrow \mathbb{K}$ on the quotient stack

$$[X/G] := \mathrm{colim}\Big(X \leftleftarrows X \times G \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; X \times G^2 \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \cdots \Big) \in \mathrm{dSt}.$$

The derived critical locus is defined as the pullback in dSt

$$\begin{tikzcd}
\mathrm{dCrit}(S) \arrow[r, dashed] \arrow[d, dashed] & {[X/G]} \arrow[d, "0"] \\
{[X/G]} \arrow[r, "d^{dR}S"'] & T^*[X/G]
\end{tikzcd}$$

This solves the intersection problem $d^{dR}S = 0$ and also counts multiplicities and stabilizers !

Theorem (Benini-Safronov-AS):

$d\mathrm{Crit}(S) \simeq [Z/G]$ is a derived quotient stack with $Z = \mathrm{Spec}\, \mathcal{O}(Z)_\bullet$ the derived affine scheme specified by the function dg-algebra

$$\mathcal{O}(Z)_\bullet = \underbrace{\mathrm{Sym}_A}_{\text{fields}} \Big(A \otimes \underbrace{\mathfrak{g}[-2]}_{\substack{\text{antifields} \\ \text{for ghosts}}} \oplus \underbrace{T_A[-1]}_{\text{antifields}} \Big) \in \mathrm{dgCAlg}_{\geq 0}$$

$$\partial a = 0 \quad , \quad \partial v = i_v(d^{dR} S) \quad , \quad \partial t = \rho^*(t) = -\, i_{\rho(t)} \lambda$$

tautological 1-form on T^*X

Remarks:

(1) $d\mathrm{Crit}(S)$ carries a canonical (-1)-shifted symplectic structure that can be computed via Lagrangian intersections.

(2) The function dg-algebra can be compared with the BV formalism:

$$\mathcal{O}(d\mathrm{Crit}(S)) \simeq \mathcal{O}([Z/G]) \simeq \mathrm{Tot}^{\Pi} \underbrace{N^\bullet(G, \mathcal{O}(Z)_\bullet)}_{\text{normalized group cochains}}$$

$$\Big\downarrow \text{van Est map}$$

! In general NOT $\simeq$

Lie algebra of G

$$\mathcal{O}(BV(S)) \simeq \mathcal{O}([Z/\mathfrak{g}]) \simeq \mathrm{Tot}^{\Pi}\, CE^\bullet(\mathfrak{g}, \mathcal{O}(Z)_\bullet)$$

(3) The derived stack $d\mathrm{Crit}(S) \simeq [Z/G]$ is in general **not** affine, i.e. it is not determined by its function dg-algebra.

This is a new feature of the nonperturbative world!

A richer algebraic invariant is given by its dg-category of modules:

$$\mathrm{QCoh}(d\mathrm{Crit}(S)) \simeq \mathrm{QCoh}([Z/G]) \simeq {}_{\mathcal{O}(Z)_\bullet}\mathrm{dgMod}^G$$

Application 2: Nonperturbative BFV quantization

- **Physical scenario:**

The phase space of 2nd-order gauge theories is a derived cotangent stack

$$\underbrace{T^*}_{\substack{\text{canonical}\\\text{momenta}}}[\underbrace{X}_{\substack{\text{position}\\\text{variables}}}/\underbrace{G}_{\substack{\text{gauge}\\\text{symmetries}}}] \overset{\text{Safronov}}{\simeq} \underbrace{[T^*X /\!/ G]}_{\substack{\text{symplectic}\\\text{reduction}}} := [\mu^{-1}(0)/G]$$

Similarly to before, $\mu^{-1}(0) = \mathrm{Spec}\,\mathcal{O}(\mu^{-1}(0))_\bullet$ is a derived affine scheme with

$$\mathcal{O}(\mu^{-1}(0))_\bullet = \mathrm{Sym}_A\big(T_A \xleftarrow{\mu^*} A\otimes \mathfrak{g}_{[-1]}\big) \in \mathrm{dgCAlg}_{\geq 0}\,.$$

- **Wanted:**

Quantization of $T^*[X/G]$ along the canonical 0-shifted Poisson structure.

How does the resulting E_0 = pointed dg-category look like:

$$\mathrm{QCoh}(T^*[X/G])_\hbar = \,?$$

- **Strategy:**

Turn Pridham's abstract deformation theoretic arguments into a concrete construction! Let me sketch the key ideas:

Step 1: Resolve $T^*[X/G] \simeq [\mu^{-1}(0)/G]$ by a diagram of Lie algebra quotients:

$$\boxed{[\mu^{-1}(0)/\mathfrak{g}]} \leftleftarrows \boxed{[\mu^{-1}(0)\times G/\mathfrak{g}\oplus\mathfrak{g}] \;\leftleftarrows\!\!\!\leftarrow \;\cdots}$$

BFV formalism + non perturbative features

This turns the global problem into a family of local stacky affine problems:

$$CE^\bullet(\mathfrak{g}, \mathcal{O}(\mu^{-1}(0))_\bullet) \rightrightarrows CE^\bullet(\mathfrak{g}\oplus\mathfrak{g}, \mathcal{O}(\mu^{-1}(0)\times G)_\bullet) \;\overset{\rightarrow}{\rightrightarrows}\; \cdots$$

Step 2: Quantize level-wise via differential operators

$$CE^\bullet(\mathfrak{g}, \mathcal{O}(\mu^{-1}(0))_\bullet)_\hbar \rightrightarrows CE^\bullet(\mathfrak{g}\oplus\mathfrak{g}, \mathcal{O}(\mu^{-1}(0)\times G)_\bullet)_\hbar \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots$$

and pass over to dg-categories of modules:

$$\circledast := \left({}_{CE^\bullet(\mathfrak{g}, \mathcal{O}(\mu^{-1}(0))_\bullet)_\hbar}\mathrm{dgMod} \rightrightarrows {}_{CE^\bullet(\mathfrak{g}\oplus\mathfrak{g}, \mathcal{O}(\mu^{-1}(0)\times G)_\bullet)_\hbar}\mathrm{dgMod} \mathrel{\substack{\rightarrow\\\rightarrow\\\rightarrow}} \cdots \right)$$

Step 3: Obtain a global quantization by computing the homotopy limit (in dgCat) of the local quantizations:

$$\mathrm{QCoh}(T^*[X/G])_\hbar := \mathrm{holim}\ \circledast \quad \in \mathrm{dgCat}.$$

Theorem (Benini-Pridham-AS):

For G reductive, the following dg-category is a model for $\mathrm{QCoh}(T^*[X/G])_\hbar$.

Objects:

Triples $(\mathcal{E}_\bullet, \nabla, \Psi)$ consisting of

(1) a G-eqv. $\mathcal{O}(X)[[\hbar]]$-dg-module $\mathcal{E}_\bullet$ (wave functions w/ G-action)

(2) a G-eqv. dg-connection $\nabla : \mathcal{E}_\bullet \to \Omega^1(X)[[\hbar]] \otimes_{\mathcal{O}(X)[[\hbar]]} \mathcal{E}_\bullet$ (action of canonical momenta)

w.r.t. $\hbar\, d^{dR}$, i.e. $\underbrace{\nabla(as) = \hbar\, d^{dR}a \otimes s + a\nabla(s)}_{\text{CCR: positions and momenta}}$

(3) a G-eqv. graded module map $\Psi : \mathfrak{g}_{[-1]} \otimes \mathcal{E}_\# \to \mathcal{E}_\#$ (action of antighosts)

These data have to satisfy the following conditions:

$$\left.\begin{array}{ll}(\mathrm{i}) & \nabla_v \nabla_{v'} - \nabla_{v'} \nabla_v = \hbar\, \nabla_{[v,v']} \\ & \nabla_v \Psi_+ - \Psi_+ \nabla_v = 0 \\ & \Psi_+ \Psi_{+'} + \Psi_{+'} \Psi_+ = 0 \end{array}\right] \text{CCR: momenta and antighosts}$$

$$(\mathrm{ii}) \quad \underbrace{\partial \Psi_+ + \Psi_+ \partial = \nabla_{\mu^*(+)} + \hbar\, g(+)}_{\text{CCR: ghosts and antighosts}}$$

(+ technical conditions)

Morphisms:

$$\underline{\mathrm{hom}}\Big((\mathcal{E}_\bullet, \nabla, \Psi), (\mathcal{E}_\bullet', \nabla', \Psi')\Big) := \underline{\mathrm{hom}}^{G,\nabla,\Psi}_{\mathcal{O}(X)[[\hbar]]}(\mathcal{E}_\bullet, \mathcal{E}_\bullet')$$

preserving the G, ∇, Ψ structures strictly.

Remark:

This result can be used to construct a dg-categorified lattice AQFT for non-Abelian Yang-Mills theory on directed graphs:

$$\mathfrak{A} : \boxed{\mathcal{P}_{\mathrm{DiGraph}}} \longrightarrow \boxed{\mathbf{dgCat}}$$

prefactorization operad over orthogonal category of directed graphs

SM category of dg-categories