From Fredenhagen's universal algebra to homotopy theory and operads

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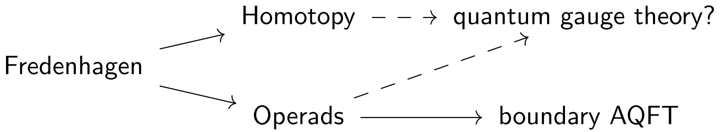
Quantum Physics meets Mathematics:
A workshop on the occasion of Klaus Fredenhagen's 70th birthday,
University of Hamburg, December 8-9, 2017.

Outline

1. Fredenhagen's universal algebra and some of its applications in AQFT
2. How it provided motivations for our homotopical AQFT program
3. A whole zoo of (improved) universal constructions from operad theory
4. **Birthday present:**

   A theorem about AQFTs on spacetimes with timelike boundary



Fredenhagen's universal algebra

Origins

◇ Starting late 80's: Klaus studied representation theory and superselection sectors of 2-dim. QFTs, including chiral conformal QFTs [cf. Carpi's talk].

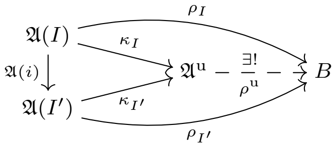
◇ A ccQFT has an underlying functor $\mathfrak{A} : \mathbf{Int}(\mathbb{S}^1) \to \mathbf{Alg}$, which assigns

- an algebra (of observables) $\mathfrak{A}(I)$ to every proper interval $I \subset \mathbb{S}^1$;
- a homomorphism $\mathfrak{A}(i) : \mathfrak{A}(I) \to \mathfrak{A}(I')$ to every inclusion $i : I \to I'$.

Def: Fredenhagen's universal algebra corresponding to $\mathfrak{A} : \mathbf{Int}(\mathbb{S}^1) \to \mathbf{Alg}$ is

- an algebra $\mathfrak{A}^{\mathrm{u}} \in \mathbf{Alg}$;
- together with homs $\kappa_I : \mathfrak{A}(I) \to \mathfrak{A}^{\mathrm{u}}$ satisfying $\kappa_{I'} \circ \mathfrak{A}(i) = \kappa_I$, for all i,

which are universal: For any other such $(B, \{\rho_I : \mathfrak{A}(I) \to B\})$ there exists a unique hom $\rho^{\mathrm{u}} : \mathfrak{A}^{\mathrm{u}} \to B$, such that the following diagrams commute



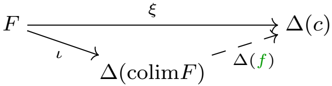
◇ $\mathfrak{A}^{\mathrm{u}}$ is useful for representation theory $\mathrm{Rep}(\mathfrak{A}) \cong \mathrm{Rep}(\mathfrak{A}^{\mathrm{u}})$.

Intermezzo: Colimits

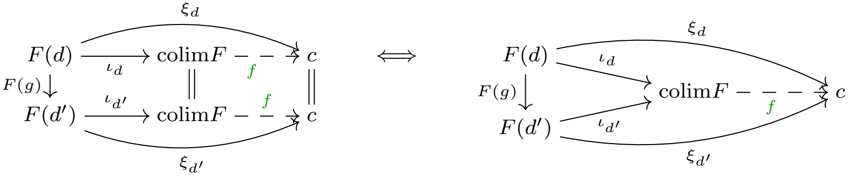
- The universal algebra is a special instance of a colimit in category theory.

Def: (i) A cocone of a functor $F : \mathbf{D} \to \mathbf{C}$ is an object $c \in \mathbf{C}$ together with a natural transformation $\xi : F \to \Delta(c)$ to the constant functor $\Delta(c) : d \mapsto c$.

(ii) A colimit of F is a universal cocone $(\mathrm{colim} F, \iota : F \to \Delta(\mathrm{colim} F))$, i.e. given any other cocone $(c, \xi : F \to \Delta(c))$ there exists a unique $\mathbf{C}$-morphism $f : \mathrm{colim} F \to c$, such that the following diagram commutes



- Fredenhagen's definition is recovered by writing this in components:



- **Good News:** For functors $\mathfrak{A} : \mathbf{D} \to \mathbf{Alg}$ with values in algebras, the colimit $\mathrm{colim}\, \mathfrak{A}$ always exists! $\Rightarrow$ Fredenhagen's universal algebra always exists!

Beyond the circle

- The universal algebra is a very flexible concept!
- **Problem:** Want to construct $\mathfrak{A}(M)$ on complicated spacetime M, but just manage to get a functor $\mathfrak{A} : \mathbf{Reg}_M \to \mathbf{Alg}$ on 'nice' regions $U \subseteq M$.
- **Solution:** Set $\mathfrak{A}(M) := \mathrm{colim}\big(\mathfrak{A} : \mathbf{Reg}_M \to \mathbf{Alg}\big)$ to be universal algebra!
- Together with students, Klaus studied particular applications:

1. **Maxwell theory** [Benni Lang, Diplomarbeit 2010]

   Maxwell's equations $\mathrm{d}F = 0 = \delta F$ for $F \in \Omega^2(M)$ allow for topological charges $[F] \in H^2(M;\mathbb{R})$ and $[*F] \in H^{m-2}(M;\mathbb{R})$ on general spacetimes M.

   Construct $\mathfrak{A} : \mathbf{Reg}_M \to \mathbf{Alg}$ for contractible regions $\mathbf{Reg}_M$ in M and analyze properties of the global algebra $\mathfrak{A}(M) := \mathrm{colim}\,\mathfrak{A}$. [more on this later...]

2. **Non-globally hyperbolic spacetimes** [Christian Sommer, Diplomarbeit 2006]

   Let M be a spacetime with timelike boundary and consider globally hyperbolic regions $\mathbf{Reg}_{\mathrm{int}M}$ in the interior $\mathrm{int}M \subseteq M$.

   Assuming F-locality [Kay], $\mathfrak{A} : \mathbf{Reg}_{\mathrm{int}M} \to \mathbf{Alg}$ can be constructed as usual.

   Relationship between ideals of $\mathfrak{A}(M) := \mathrm{colim}\,\mathfrak{A}$ and boundary conditions!

Fredenhagen's universal algebra in LCQFT

◇ In locally covariant QFT, one studies functors $\mathfrak{A} : \mathbf{Loc} \to \mathbf{Alg}$ on the category of all spacetimes $\mathbf{Loc}$ [cf. Fewster's talk].

◇ Consider full subcategory $j : \mathbf{Loc}_{©} \xrightarrow{\subseteq} \mathbf{Loc}$ of contractible spacetimes and assume that you are given a theory $\mathfrak{A}_{©} : \mathbf{Loc}_{©} \to \mathbf{Alg}$.

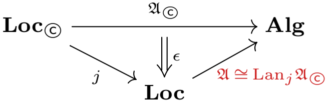
◇ **Construction/Observation:** [Benni Lang, PhD in York (2014) with Fewster]

- On every spacetime $M \in \mathbf{Loc}$, we may compute the universal algebra

$$\mathfrak{A}(M) := \mathrm{colim}\Big(\mathbf{Loc}_{©}/M \xrightarrow{Q_M} \mathbf{Loc}_{©} \xrightarrow{\mathfrak{A}_{©}} \mathbf{Alg} \Big)$$

  on the over category $\mathbf{Loc}_{©}/M$ of contractible regions $U \to M$ in M.

- This defines functor $\mathfrak{A} : \mathbf{Loc} \to \mathbf{Alg}$ on all spacetimes $\mathbf{Loc}$, which is universal in the sense of left Kan extensions



Universal algebra in LCQFT = left Kan extension along $j : \mathbf{Loc}_{©} \to \mathbf{Loc}$

The homotopical AQFT program

M. Benini, AS, U. Schreiber, R. J. Szabo and L. Woike

Universal algebra for Maxwell theory

◇ Classical Maxwell theory on contractible spacetimes $U \in \mathbf{Loc}_{©}$:

- $A \in \Omega^1(U)$ with gauge trafos $A \mapsto A + \frac{1}{2\pi\mathrm{i}}\mathrm{d}\log g$, for $g \in C^\infty(U, U(1))$
- Maxwell's equation $\delta\mathrm{d}A = 0$, i.e. $A \in \Omega^1_{\delta\mathrm{d}}(U)$

◇ Gauge invariant and on-shell exponential observables $[\varphi] \in \Omega^1_{\mathrm{c}\,\delta}(U)\big/\delta\mathrm{d}\Omega^1_{\mathrm{c}}(U)$

$$\mathcal{O}_{[\varphi]} \;:\; \frac{\Omega^1_{\delta\mathrm{d}}(U)}{\mathrm{d}\Omega^0(U)} \longrightarrow \mathbb{C}\;, \quad [A] \longmapsto \exp\Big(2\pi\mathrm{i}\int_U \varphi \wedge *A\Big)$$

with presymplectic structure $\omega_U([\varphi],[\varphi']) = \exp\big(2\pi\mathrm{i}\int_U \varphi \wedge *G_{\Box}(\varphi')\big)$.

◇ Quantum Maxwell theory $\mathfrak{A}_{©} : \mathbf{Loc}_{©} \to \mathbf{Alg}$ assigns the Weyl algebras.

◇ Universal algebra $\mathfrak{A}(M)$ is the Weyl algebra corresponding to field strength theory on $M \in \mathbf{Loc}$ [Dappiaggi,Lang]: $F \in \Omega^2(M)$ satisfying $\delta F = 0 = \mathrm{d}F$

◇ **Problem:** $\mathfrak{A}(M)$ does **NOT** have a gauge theoretic interpretation

1. It misses flat connections/Aharonov-Bohm phases on M!
2. $[F] \in H^2(M;\mathbb{R})$ is not integral $\Rightarrow$ No magnetic charge quantization!

Homotopical improvement of the universal algebra

- **Important lesson:** Do **NOT** quotient out the gauge symmetries naively!
- Chain complex of $U(1)$-gauge fields on $U \in \mathbf{Loc}_{©}$

$$\mathcal{F}_{©}(U) \;:=\; \Big(\, \Omega^1(U) \xleftarrow{\frac{1}{2\pi\mathrm{i}}\mathrm{d}\log} C^\infty(U,U(1)) \,\Big)$$

- Smooth Pontryagin dual chain complex of observables on $U \in \mathbf{Loc}_{©}$

$$\mathcal{O}_{©}(U) \;:=\; \Big(\, \Omega^{m-1}_{\mathrm{c}}(U) \xrightarrow{\;\mathrm{d}\;} \Omega^m_{\mathrm{c};\mathbb{Z}}(U) \,\Big)$$

- Extension to $M \in \mathbf{Loc}$ via homotopy colimit/homotopy left Kan extension

$$\mathcal{O}(M) \;:=\; \mathrm{hocolim}\Big(\, \mathbf{Loc}_{©}/M \xrightarrow{Q_M} \mathbf{Loc}_{©} \xrightarrow{\mathcal{O}_{©}} \mathbf{Ch}(\mathbf{Ab}) \,\Big)$$

Theorem [Benini,AS,Szabo]

For every $M \in \mathbf{Loc}$, $\mathcal{O}(M)$ is weakly equivalent to dual Deligne complex on M.

In particular, it contains observables for flat connections and respects magnetic charge quantization of gauge theories!

Bird's-eye view on homotopical AQFT

◇ **Higher structures in gauge theory:** [cf. Gwilliam's talk]

- 'Spaces' of gauge fields are not usual spaces, but higher spaces called stacks.
- Consequently, observable 'algebras' for gauge theories are not conventional algebras, but higher algebras, e.g. differential graded algebras.

◇ To formalize quantum gauge theories, we develop

homotopical AQFT := AQFT + homotopical algebra

'Def:' A homotopical AQFT is an assignment $\mathfrak{A} : \mathbf{Loc} \to \mathbf{dgAlg}$ of differential graded algebras (or other higher algebras) to spacetimes, satisfying

1. functoriality, causality and time-slice (possibly up to coherent homotopies);
2. local-to-global property, i.e. $\mathfrak{A}$ is homotopy left Kan extension of $\mathfrak{A}|_{\mathbf{Loc}_{©}}$.

◇ **Our results:**

- Global observables via homotopy left Kan extension [Benini,AS,Szabo]
- Toy-models via orbifoldization (homotopy invariants) [Benini,AS]
- Yang-Mills stack and stacky Cauchy problem [Benini,AS,Schreiber]
- Towards a precise definition in terms of operads [Benini,AS,Woike]

NB: BRST/BV formalism of [Fredenhagen,Rejzner] should provide examples.

The operadic AQFT program

M. Benini, S. Bruinsma, AS and L. Woike

Categories of AQFTs: General perspective

◇ **Input data:** (so that we can talk about QFTs)

- Category $\mathbf{C}$ ('spacetimes') with subset $W \subseteq \mathrm{Mor}\mathbf{C}$ ('Cauchy morphisms')
- Orthogonality $\perp \subseteq \mathrm{Mor}\mathbf{C}_t \times_t \mathrm{Mor}\mathbf{C}$ ('causally disjoint' $(c_1 \to c \leftarrow c_2) \in \perp$)
- Target category $\mathbf{M}$ (bicomplete closed symmetric monoidal)

Def: The category of $\mathbf{M}$-valued AQFTs on $(\mathbf{C}, W, \perp)$ is the full subcategory $\mathbf{qft}(\mathbf{C}, W, \perp) \subseteq \mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}}$ of functors $\mathfrak{A} : \mathbf{C} \to \mathbf{Mon}_{\mathbf{M}}$ satisfying

1. W-constancy: For all $f \in W$, $\mathfrak{A}(f)$ is isomorphism.
2. $\perp$-commutativity: For all $f_1 \perp f_2$, the following diagram commutes

$$\begin{array}{ccc}
\mathfrak{A}(c_1) \otimes \mathfrak{A}(c_2) & \xrightarrow{\mathfrak{A}(f_1)\otimes\mathfrak{A}(f_2)} & \mathfrak{A}(c) \otimes \mathfrak{A}(c) \\
{\scriptstyle \mathfrak{A}(f_1)\otimes\mathfrak{A}(f_2)}\downarrow & & \downarrow{\scriptstyle \mu_c^{\mathrm{op}}} \\
\mathfrak{A}(c) \otimes \mathfrak{A}(c) & \xrightarrow[\mu_c]{} & \mathfrak{A}(c)
\end{array}$$

Thm: Localization induces equivalence $\mathbf{qft}(\mathbf{C}, W, \perp) \cong \mathbf{qft}(\mathbf{C}[W^{-1}], \emptyset, L_*(\perp))$.

! This means that W-constancy can be hard-coded as a structure.

NB: The relevant categories are $\mathbf{QFT}(\mathbf{C}, \perp) := \mathbf{qft}(\mathbf{C}, \emptyset, \perp)$.

AQFTs are algebras over a colored operad

- ◇ Operads capture abstractly the operations underlying algebraic structures.
- ◇ **Example:** Associative and unital algebras

$$\mathrm{Ass}\text{-operad} \xrightarrow{\text{operad algebras}} \text{abstract algebras} \xrightarrow{\text{representations}} \text{concrete algebras}$$

$\mathrm{Ass}(n) = \{\mu_n\}$, for $n \geq 0$ $\qquad \mu_n^A : A^{\otimes n} \to A \qquad \mu_n^{\mathrm{End}(V)} : \mathrm{End}(V)^{\otimes n} \to \mathrm{End}(V)$

Theorem [Benini,AS,Woike]

For every $(\mathbf{C}, \perp)$, there exists an $\mathrm{Ob}(\mathbf{C})$-colored operad $\mathcal{O}_{(\mathbf{C},\perp)}$ whose category of algebras is canonically isomorphic to the category of AQFTs on $(\mathbf{C}, \perp)$, i.e.

$$\mathbf{Alg}(\mathcal{O}_{(\mathbf{C},\perp)}) \cong \mathbf{QFT}(\mathbf{C}, \perp)$$

! This means that $\perp$-commutativity can be hard-coded as a structure by using colored operads. $\rightsquigarrow$ Very useful for universal constructions, see next slide.

Rem: Precise operadic definition of homotopical AQFT:

$$\textbf{homotopical AQFT} := \mathcal{O}_{(\mathbf{C},\perp)\,\infty}\textbf{-algebra} + \textbf{local-to-global property}$$

A whole zoo of universal constructions

- **Main observation:** The assignment of AQFT operads $(\mathbf{C},\perp) \mapsto \mathcal{O}_{(\mathbf{C},\perp)}$ is functorial $\mathcal{O} : \mathbf{OrthCat} \to \mathbf{Op}(\mathbf{M})$ on the category of orthogonal categories.

$\Rightarrow$ For every orthogonal functor $F : (\mathbf{C},\perp_{\mathbf{C}}) \to (\mathbf{D},\perp_{\mathbf{D}})$ we obtain adjunction

$$\mathbf{QFT}(\mathbf{C},\perp_{\mathbf{C}}) \underset{F^*}{\overset{F_!}{\rightleftarrows}} \mathbf{QFT}(\mathbf{D},\perp_{\mathbf{D}})$$

! Because the W-constancy and $\perp$-commutativity axioms are hard-coded as structures in our operads, these adjunctions always produce AQFTs.

- **Examples:**

1. $\perp$-Abelianization: $\mathrm{id}_{\mathbf{C}} : (\mathbf{C},\emptyset) \to (\mathbf{C},\perp)$ induces

$$\mathbf{Mon}_{\mathbf{M}}^{\mathbf{C}} \underset{\text{Forget}}{\overset{\text{Ab}}{\rightleftarrows}} \mathbf{QFT}(\mathbf{C},\perp)$$

2. W-constantification: $L : (\mathbf{C},\perp) \to (\mathbf{C}[W^{-1}], L_*(\perp))$ induces

$$\mathbf{QFT}(\mathbf{C},\perp) \underset{L^*}{\overset{L_!}{\rightleftarrows}} \mathbf{QFT}(\mathbf{C}[W^{-1}], L_*(\perp))$$

3. Local-to-global: Full orthogonal subcat $j : (\mathbf{C}, j^*(\perp)) \to (\mathbf{D},\perp)$ induces

$$\mathbf{QFT}(\mathbf{C}, j^*(\perp)) \underset{j^*}{\overset{j_!}{\rightleftarrows}} \mathbf{QFT}(\mathbf{D},\perp)$$

Comparison to Fredenhagen's universal algebra

◇ Fredenhagen's universal algebra ignores the $\perp$-commutativity axiom.

◇ Comparison via diagram of adjunctions (square of right adjoints commutes)

$$\begin{array}{ccc}
\mathbf{QFT}(\mathbf{Loc}_{©}, j^*(\perp)) & \underset{j^*}{\overset{j_!}{\rightleftarrows}} & \mathbf{QFT}(\mathbf{Loc}, \perp) \\
\mathrm{Ab} \uparrow \downarrow \mathrm{Forget} & & \mathrm{Ab} \uparrow \downarrow \mathrm{Forget} \\
\mathbf{Mon}_{\mathbf{M}}^{\mathbf{Loc}_{©}} & \underset{j^*}{\overset{\mathrm{Lan}_j}{\rightleftarrows}} & \mathbf{Mon}_{\mathbf{M}}^{\mathbf{Loc}}
\end{array}$$

(operadic)

⇝ forget $\perp$

(Fredenhagen)

Theorem [Benini,AS,Woike]

1. Let $\mathfrak{A} \in \mathbf{QFT}(\mathbf{Loc}_{©}, j^*(\perp))$ be such that $\mathrm{Lan}_j \mathrm{Forget}\, \mathfrak{A}$ is $\perp$-commutative. Then $\mathrm{Lan}_j \mathrm{Forget}\, \mathfrak{A} \cong \mathrm{Forget}\, j_!\, \mathfrak{A}$.
2. $\mathrm{Lan}_j \mathrm{Forget}\, \mathfrak{A}$ is $\perp$-commutative over $M \in \mathbf{Loc}$, for all $\mathfrak{A}$, **if and only if** for all causally disjoint $f_1 : U_1 \to M \leftarrow U_2 : f_2$ with $U_1, U_2 \in \mathbf{Loc}_{©}$ there exists

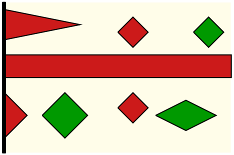
$$\begin{array}{ccccc}
 & & M & & \\
 & \overset{f_1}{\nearrow} & \uparrow & \overset{f_2}{\nwarrow} & \\
U_1 & \dashrightarrow & U & \dashleftarrow & U_2
\end{array}$$

Rem: Property in 2. violated for disconnected M. Open question: Connected M?

A characterization theorem for boundary AQFTs

M. Benini, C. Dappiaggi and AS (to appear soon)

Universal boundary extensions of AQFTs



- ◇ Spacetime M with timelike boundary
- ◇ $\mathbf{QFT}(\mathrm{int}M)$ on causally compatible interior regions
- ◇ $\mathbf{QFT}(M)$ on **all** causally compatible regions
- ◇ **Universal boundary extension** (no choice of boundary conditions needed!)

$$\mathbf{QFT}(\mathrm{int}M) \underset{\mathrm{res}}{\overset{\mathrm{ext}}{\rightleftarrows}} \mathbf{QFT}(M)$$

- ◇ Given $\mathfrak{B} \in \mathbf{QFT}(M)$, the counit of the adjunction provides comparison map

$$\epsilon_{\mathfrak{B}} \,:\, \mathrm{ext}\,\mathrm{res}\,\mathfrak{B} \longrightarrow \mathfrak{B}$$

Theorem [Benini,Dappiaggi,AS]

$\epsilon_{\mathfrak{B}} : \mathrm{ext}\,\mathrm{res}\,\mathfrak{B}/\ker\epsilon_{\mathfrak{B}} \to \mathfrak{B}$ is isomorphism **if and only if** $\mathfrak{B}$ is generated from the interior, i.e. every $\mathfrak{B}(V)$ is generated by $\mathfrak{B}(V_{\mathrm{int}})$, for all interior regions $V_{\mathrm{int}} \subseteq V$.

Rem: Implies that every such $\mathfrak{B}$ may be described by two independent data:

1. A theory on the interior $\mathfrak{A} \in \mathbf{QFT}(\mathrm{int}M)$
2. An ideal $\mathfrak{I} \subseteq \mathrm{ext}\,\mathfrak{A}$ that vanishes on the interior $\Leftarrow$ boundary conditions

Happy Birthday, Klaus!

Klaus secretly doing homotopy theory in the black forest!

(MFO Mini-Workshop: New interactions between homotopical algebra and quantum field theory)