C^*-categorical prefactorization algebras for superselection sectors and topological order

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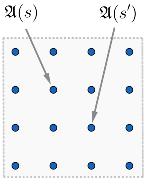
Joint work with M. Benini, V. Carmona and P. Naaijkens [arXiv:2505.07960].

Lattice quantum systems and topological excitations

◇ Fix a discrete set S, e.g. $S = \mathbb{Z}^n$. By a lattice quantum system I mean a family of C^*-algebras $\{\mathfrak{A}(s)\}_{s \in S}$ and a ground state $\omega_0 : \mathfrak{A}(S) \to \mathbb{C}$.



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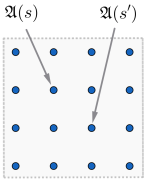

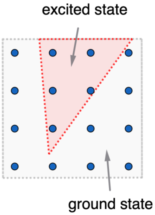

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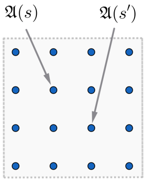

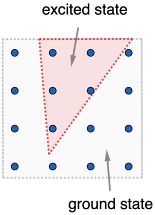

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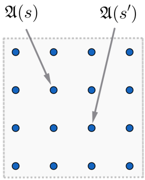

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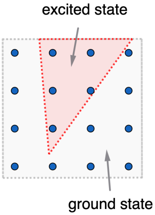

- ◇ An interesting feature of some lattice quantum systems (e.g. Kitaev model) is that these topological excitations assemble into a braided monoidal category.

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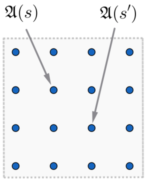

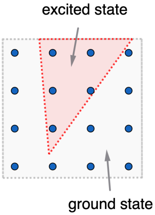

Goal of this talk:

Provide a conceptual and geometric explanation of this story from a novel point of view combining AQFT, factorization algebras and higher algebra.

From lattice quantum systems to AQFTs

- One can extend a lattice quantum system to an AQFT over S, i.e. a functor $\mathfrak{A} : \mathbf{Sub}(S) \to C^*\mathbf{Alg}$ satisfying locality on disjoint subsets

$$[-,-]_{\mathfrak{A}(V)} \circ \big(\mathfrak{A}(\iota_{U_1}^V) \otimes \mathfrak{A}(\iota_{U_2}^V)\big) \,=\, 0 \quad , \quad \text{for all } (U_1 \subseteq V) \perp (U_2 \subseteq V) \quad .$$

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- Abstractly, this is described by a fully faithful operadic left Kan extension

$$j_! \,:\, \textstyle\prod_{s\in S} C^*\mathbf{Alg} \rightleftarrows \mathbf{AQFT}(S) \,:\, j^*$$

along the singleton set inclusion functor $j : S \to \mathbf{Sub}(S)\,,~ s \mapsto \{s\}$.

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- This gives a nice and simple model for the representation C^*-category

$$\mathbf{Rep}_{\mathfrak{A}} := \mathbf{Rep}_{\mathfrak{A}(S)} \simeq \begin{cases} \text{Obj:} & \big\{\pi_s : \mathfrak{A}(s) \to B(H)\big\}_{s\in S} \text{ Hilbert space reps} \\ & \text{s.t. } [-,-]_{B(H)} \circ (\pi_s \otimes \pi_{s'}) = 0 \ \ \forall s \neq s' \\ \text{Mor:} & T : H \to H' \text{ bounded operator} \\ & \text{s.t. } T\,\pi_s(-) = \pi'_s(-)\,T \ \ \forall s \end{cases}$$

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- The ground state ω_0 defines via GNS a distinguished object $\pi_0 \in \mathbf{Rep}_{\mathfrak{A}}$.

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- The basic idea is simple: Consider special types of representations $\pi \in \mathbf{Rep}_{\mathfrak{A}}$ which are unitarily equivalent to the ground state

$$\pi|_{\mathfrak{A}(U^{\mathrm{c}})} \cong \pi_0|_{\mathfrak{A}(U^{\mathrm{c}})} \quad , \quad \text{for } \underline{\text{all}}\ U \in \mathbf{C}(S) \quad ,$$

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- Up to equivalence of C^*-categories, one can further demand w.l.o.g. for any <u>fixed</u> $U \in \mathbf{C}(S)$ the strict localization condition $\pi|_{\mathfrak{A}(U^c)} = \pi_0|_{\mathfrak{A}(U^c)}$.

  Denote by $\mathbf{SSS}(U) \subseteq \mathbf{Rep}_{\mathfrak{A}}$ the full C^*-subcategory of superselection sectors that are also strictly localized in $U \in \mathbf{C}(S)$.

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Lem: These C^*-categories assemble into a functor

$$\mathbf{SSS} \, : \, \mathbf{C}(S) \longrightarrow C^*\mathbf{Cat}$$

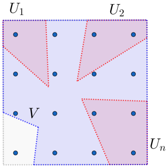
which is locally constant (or 'topological') in the sense that it assigns to every morphism $U \subseteq V$ in $\mathbf{C}(S)$ a unitary equivalence $\mathbf{SSS}(U) \xrightarrow{\sim} \mathbf{SSS}(V)$.

Factorization products: Candidates

- Would like to endow the functor $\mathbf{SSS} : \mathbf{C}(S) \to C^*\mathbf{Cat}$ with an additional multiplicative structure

$$\bullet \, : \, \bigotimes_{i=1}^{n} \mathbf{SSS}(U_i) \,\longrightarrow\, \mathbf{SSS}(V)$$

for mutually disjoint inclusions $(U_1, \dots, U_n) \to V$.



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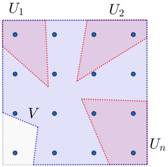

- Due to the point-wise description of representations $\pi = \{\pi_s\}_{s \in S}$, there is an obvious candidate for such factorization products

$$(\pi_1 \bullet \dots \bullet \pi_n)_s \;:=\; \begin{cases} (\pi_i)_s & , \quad \text{for } s \in U_i \quad , \\ (\pi_0)_s & , \quad \text{for } s \in U_1^{\mathrm{c}} \cap \dots \cap U_n^{\mathrm{c}} \quad . \end{cases}$$

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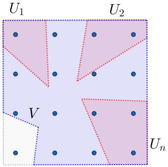

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⚠ Without additional hypotheses, these factorization products are in general not well-defined. For example, there is no reason why $(\pi_1 \bullet \cdots \bullet \pi_n)_s$ and $(\pi_1 \bullet \cdots \bullet \pi_n)_{s'}$ should be commuting for all $s \neq s'$.

Factorization products: Hypotheses

◇ **Geometric assumptions on localization regions $\mathbf{C}(S)$:**

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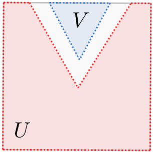

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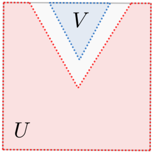

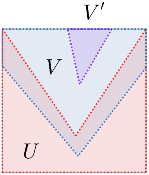

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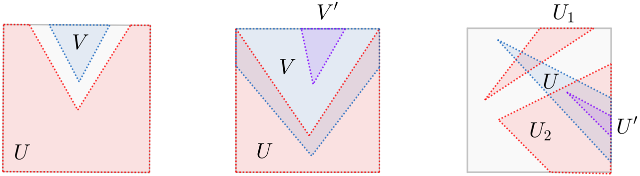

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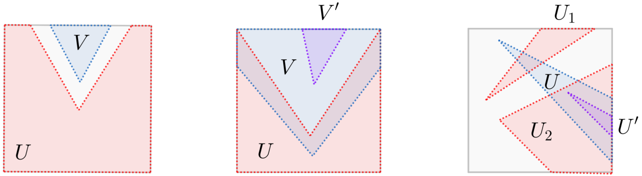

◇ **Algebraic assumptions on the AQFT $\mathfrak{A}$ and the ground state π_0:**

For each $U \in \mathbf{C}(S)$, locality gives inclusion $\pi_0(\mathfrak{A}(U^{\mathrm{c}})) \subseteq \pi_0(\mathfrak{A}(U))'$ in the commutant, which gives $\pi_0(\mathfrak{A}(U))'' \subseteq \pi_0(\mathfrak{A}(U^{\mathrm{c}}))'$ for the bicommutant.

We assume that the latter are equalities $\pi_0(\mathfrak{A}(U))'' = \pi_0(\mathfrak{A}(U^{\mathrm{c}}))'$, for all $U \in \mathbf{C}(S)$, which in AQFT is called Haag duality.

Factorization products: Results

Thm: Suppose that $\mathbf{C}(S)$ satisfies the geometric assumptions and $(\mathfrak{A}, \pi_0)$ satisfies Haag duality. Then the candidate factorization products are well-defined and define a $C^*\mathbf{Cat}$-valued locally constant prefactorization algebra on $\mathbf{C}(S)$

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Thm: The prefactorization algebra and object-wise monoidal structures are compatible in the sense that they define a locally constant prefactorization algebra with values in (strict) monoidal C^*-categories

$$\mathbf{SSS} \,:\, \mathcal{P}_{\mathbf{C}(S)} \longrightarrow \mathbf{Alg}_{\mathsf{uAs}}(C^*\mathbf{Cat}) \quad .$$

Where is the usual braided monoidal category?

- **Key point:** Prefactorization algebras are quite general objects, but in simple geometric contexts they admit more familiar descriptions. For example:

$$\Big\{\text{l.c. PFAs on disks in } \mathbb{R}^n\Big\} \quad \underset{\text{[Lurie]}}{\overset{\simeq}{\Longleftrightarrow}} \quad \Big\{\mathbb{E}_n\text{-algebras}\Big\}$$

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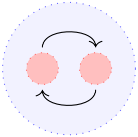

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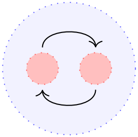

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Goal: Focus on the geometric context $\mathbf{C}(S) = \mathbf{Cone}(\mathbb{Z}^n)$ given by cones in the lattice $S = \mathbb{Z}^n$ and find out if there is a more familiar description of our

$$\mathbf{SSS} \in \mathcal{A}\mathsf{lg}^{\mathrm{l.c.}}_{\mathcal{P}_{\mathbf{Cone}(\mathbb{Z}^n)}}\Big(\mathcal{A}\mathsf{lg}_{\mathsf{uAs}}\big(C^*\mathcal{C}\mathsf{at}\big)\Big) \quad .$$

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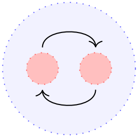

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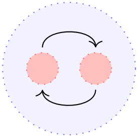
Where is the usual braided monoidal category?

- **Key point:** Prefactorization algebras are quite general objects, but in simple geometric contexts they admit more familiar descriptions. For example:



$$\big\{\text{l.c. PFAs on disks in } \mathbb{R}^n\big\} \xLeftrightarrow[\text{[Lurie]}]{\simeq} \big\{\mathbb{E}_n\text{-algebras}\big\}$$

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Note: $C^*\mathbf{Cat}$ is a combinatorial simplicial symmetric monoidal model category [Dell'Ambrogio; Bunke], so there is an associated symmetric monoidal ∞-category $C^*\mathcal{C}\mathsf{at}$.

Identifying the algebraic structures

- **Step 1:** (From cones in $\mathbb{Z}^n$ to disks in $\mathbb{S}^{n-1}$)

  Denoting by $\mathbf{Cone}_0(\mathbb{Z}^n)$ the cones with apex $0 \in \mathbb{Z}^n$, we have operad maps

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  which induce pullback ∞-functors

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Rem: The sphere $\mathbb{S}^{n-1}$ arises geometrically as the angular coordinates of cones, while the extra dimension $\mathbb{R}^1$ arises analytically from Haag duality!

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- Removing any point of the sphere $\mathbb{S}^{n-1} \setminus \{p\} \cong \mathbb{R}^{n-1}$, we get an operad map $\mathcal{P}_{\mathbf{Disk}(\mathbb{R}^n)} \to \mathcal{P}_{\mathbf{Disk}(\mathbb{R}^1 \times \mathbb{S}^{n-1})}$ and hence a (forgetful!) ∞-functor

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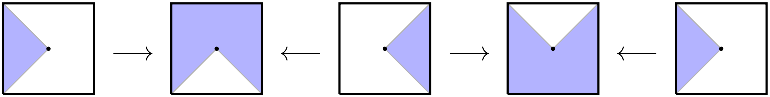
For example, in $n = 2$ dimensions one obtains a braided monoidal self-equivalence $\underline{\underline{\mathbf{SSS}}} \xrightarrow{\sim} \underline{\underline{\mathbf{SSS}}}$ as a holonomy around the circle $\mathbb{S}^1$

 .

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Thanks for listening!