Factorization Algebras vs Algebraic QFT

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Joint work with M. Benini and M. Perin [arXiv:1903.03396].

Zoology of mathematical QFT

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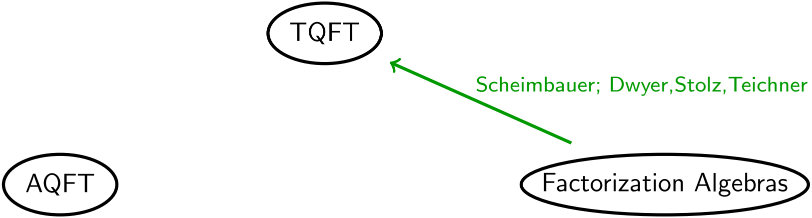

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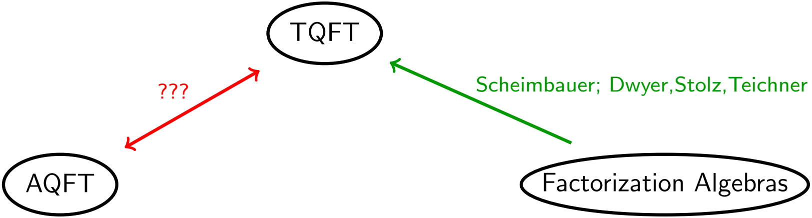

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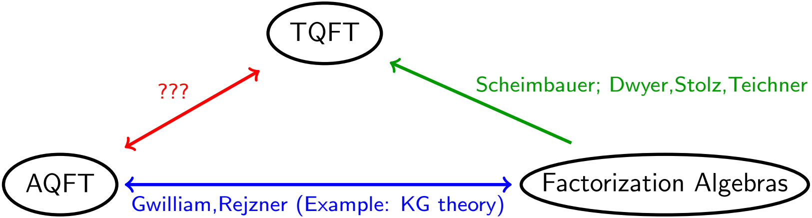

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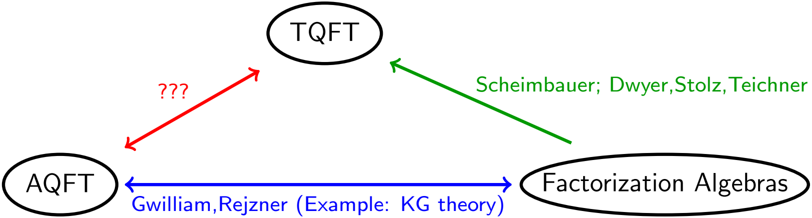

Theorem (Benini,Perin,AS)

There exists an equivalence

$$\mathbf{tPFA}^{\mathrm{add,c}} \underset{\sim}{\rightleftarrows} \mathbf{AQFT}^{\mathrm{add,c}}$$

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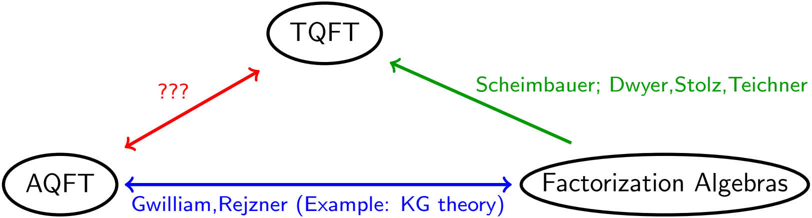

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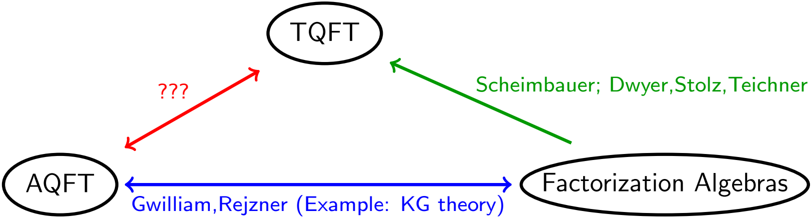

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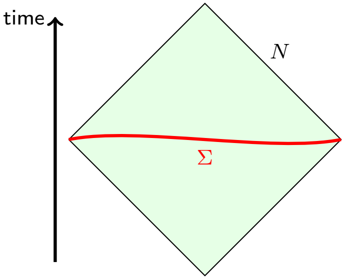
  2. Sketch the key ingredients for its proof

Some background on spacetimes

◇ **Spacetime** := oriented and time-oriented globally hyperbolic Lorentzian manifold N



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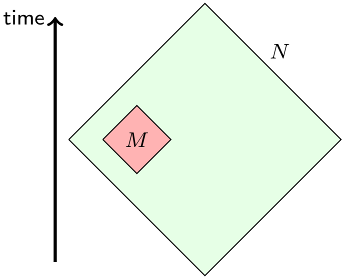

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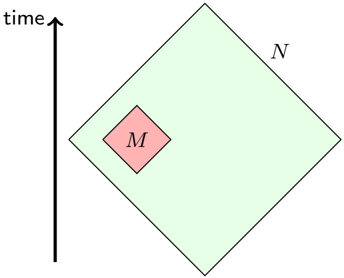

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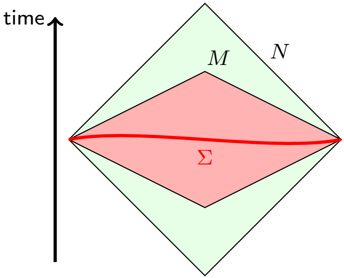

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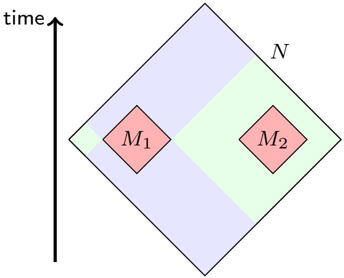

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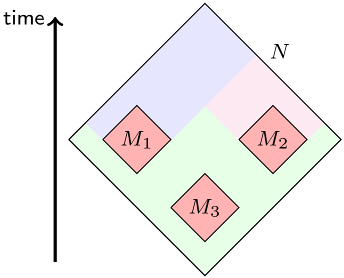

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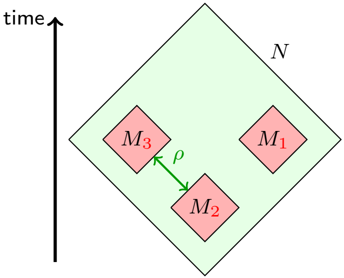

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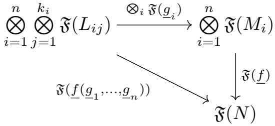

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Concretely, a morphism $\zeta : \mathfrak{F} \to \mathfrak{G}$ is a family of linear maps $\zeta_M : \mathfrak{F}(M) \to \mathfrak{G}(M)$, for all $M \in \mathbf{Loc}$, that is compatible with the factorization products in the sense that $\zeta_N \circ \mathfrak{F}(\underline{f}) = \mathfrak{F}(\underline{f}) \circ \bigotimes_i \zeta_{M_i}$, for all $\underline{f} : \underline{M} \to N$.

Further natural hypotheses on tPFAs

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$\mathfrak{F} \in \mathbf{tPFA}$ is called additive if

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Prop: Every factorization algebra $\mathfrak{F}$ on $\mathbf{Loc}$ (i.e. a tPFA satisfying Weiss descent) is an additive tPFA.

Algebraic quantum field theories on $\mathbf{Loc}$

◇ An AQFT on $\mathbf{Loc}$ is a functor $\mathfrak{A} : \mathbf{Loc} \to \mathbf{Alg} := \mathbf{Alg}_{\mathsf{As}}(\mathbf{Vec})$ satisfying the Einstein causality axiom: For causally disjoint $(f_1 : M_1 \to N, f_2 : M_2 \to N)$,

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  Slogan: Any two spacelike separated observables commute with each other.

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. . . and this now sets the stage for our Comparison Theorem.

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By abstract non-sense (operadic left Kan extension), there exists adjunction

$$\Phi_! \, : \, \mathbf{tPFA} \rightleftarrows \mathbf{AQFT} \, : \, \Phi^* = \mathfrak{F}_{(-)}$$

which however is *not* an adjoint equivalence.

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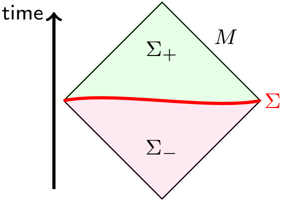

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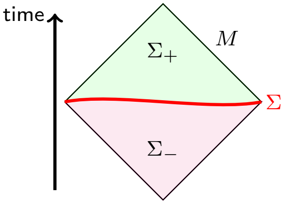

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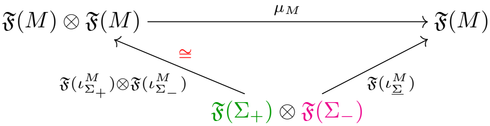

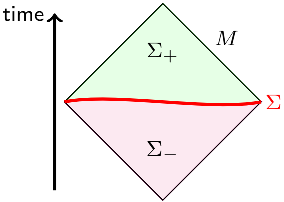

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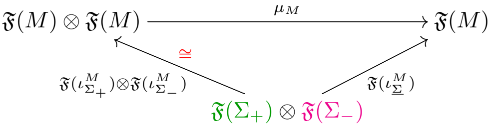

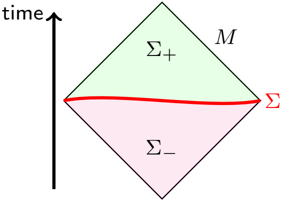

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Def: For $M \in \mathbf{Loc}$, denote by $\mathbf{P}_M$ the category of all pairs $U_\pm \subseteq M$ of causally convex open subsets fulfilling the requirements:

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Rem: This step does not yet require the additivity property for $\mathfrak{F}$, but it crucially relies on Cauchy constancy.

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Lem: Let $\mathfrak{F} \in \mathbf{tPFA}^{\mathrm{c}}$ and $f : M \to N$ be $\mathbf{Loc}$-morphism s.t. $f(M) \subseteq N$ is relatively compact. Then $\mathfrak{F}(f) : \mathfrak{F}(M) \to \mathfrak{F}(N)$ preserves units and multiplications, i.e. $\mathfrak{F}(f) \circ \eta_M = \eta_N$ and $\mathfrak{F}(f) \circ \mu_M = \mu_N \circ (\mathfrak{F}(f) \otimes \mathfrak{F}(f))$.

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Question 3: Einstein causality

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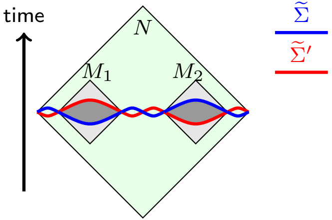
Rem: Again, the relatively compact assumption is crucial to extend Cauchy surfaces. The key step to prove Einstein causality is to find two Cauchy surfaces of N with opposite time-order when restricted to M_1 and M_2.



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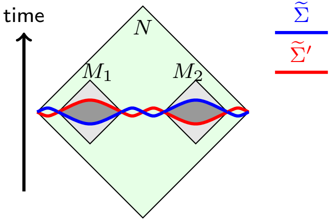

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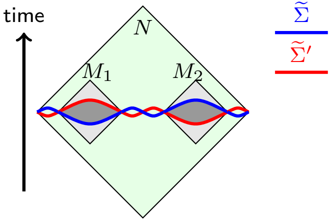

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Summary of the Main Equivalence Theorem

Theorem (Benini,Perin,AS)

The two functors

- ◇ $\mathfrak{F}_{(-)} : \mathbf{AQFT}^{\mathrm{add,c}} \to \mathbf{tPFA}^{\mathrm{add,c}}$, *and*
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described before are inverses of each other. Hence, they define an equivalence

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between the category of Cauchy constant additive time-orderable prefactorization algebras on $\mathbf{Loc}$ *and the category of Cauchy constant additive AQFTs on* $\mathbf{Loc}$.

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Thanks for your attention!