Abelian quantum gauge theories via differential cohomology

Alexander Schenkel

Department of Mathematics, Heriot-Watt University, Edinburgh

Talk @ the workshop *Algebraic quantum field theory: its status and its future*

May 19–23, 2014, ESI Vienna

work in progress with Christian Becker and Richard J. Szabo

Motivation

Part of our motivation comes from the following questions:

◇ Why doesn't gauge theory fit into the axioms of LCQFT?

– Observation originally due to [Dappiaggi,Lang] for F-theory:
$\exists\ f : M \to N$ s.t. $\mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(N)$ **not** injective

– Physical explanation due to [Dappiaggi,Hack,Sanders] for A-theory:
"electric charges" mess up the locality axiom (see Ko Sanders' talk)

– Mathematical explanation due to [Benini,Dappiaggi,Hack,AS] for ∇-theory:
QFT functor $\mathfrak{A}$ has a purely topological subfunctor related to $H^2_0(\,\cdot\,;\mathbb{R})$

◇ Is it only due to "electric charges" that locality is violated or do "magnetic charges" behave similar?

– To answer this question we have to understand what physicists mean by formulas like this:

$$\text{“}\sum_{P\in\{\text{all } \mathbb{T}\text{-bundles}\}} \int_{\mathrm{Con}(P)} D\nabla \cdots \text{”}$$

! **Differential cohomology** allows us to address these questions for Maxwell theory and also its higher versions like connections on p-gerbes!

Outline

1. What is differential cohomology?
2. Generalized Maxwell maps and solution subgroups
3. Character groups and smooth Pontryagin duals
4. Presymplectic structure on smooth Pontryagin duals
5. Quantization and properties of the QFT
6. Conclusion

What is differential cohomology?

From bundle-connection pairs to differential cohomology I

- Degrees of freedom in Maxwell theory on a manifold M are pairs (P,∇), where $P \to M$ is hermitian line bundle and ∇ hermitian connection on P.
- The set $\mathrm{Conf}(M) := \{(P,\nabla)\}$ of all such pairs is an Abelian group

$$(P,\nabla) + (P',\nabla') := (P \otimes P', \nabla \oplus \nabla') \ .$$

- **Gauge equivalence:** $(P,\nabla) \sim (P',\nabla')$ if $\exists\ f : P \to P'$ bundle isomorphism over id_M preserving the connections.
- The gauge orbit space is then the quotient $\widehat{H}^2(M;\mathbb{Z}) := \mathrm{Conf}(M)/\sim$, which is also Abelian group.
- There are natural group epimorphisms:
  - $\mathrm{curv} : \widehat{H}^2(M;\mathbb{Z}) \to \Omega^2_{\mathbb{Z}}(M) \ , \quad [(P,\nabla)] \mapsto -\frac{1}{2\pi i} R^\nabla$ (curvature)
  - $\mathrm{char} : \widehat{H}^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}) \ , \quad [(P,\nabla)] \mapsto c_1(P)$ (1^{st} Chern class)
- These maps have kernels!
  - $\ker(\mathrm{curv}) =$ flat connections $\simeq H^1(M;\mathbb{T})$
  - $\ker(\mathrm{char}) =$ eqv. classes of conn. on trivial bundle $\simeq \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$

From bundle-connection pairs to differential cohomology II

One finds that the gauge orbit space $\widehat{H}^2(M;\mathbb{Z})$ fits into the following commuting diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^1(M;\mathbb{R})}{H^1_{\mathrm{free}}(M;\mathbb{Z})} & \longrightarrow & \frac{\Omega^1(M)}{\Omega^1_{\mathbb{Z}}(M)} & \xrightarrow{\mathrm{d}} & \mathrm{d}\Omega^1(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow & & \\
0 & \longrightarrow & H^1(M;\mathbb{T}) & \xrightarrow{\kappa} & \widehat{H}^2(M;\mathbb{Z}) & \xrightarrow{\mathrm{curv}} & \Omega^2_{\mathbb{Z}}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{char}} & & \downarrow & & \\
0 & \longrightarrow & H^2_{\mathrm{tor}}(M;\mathbb{Z}) & \longrightarrow & H^2(M;\mathbb{Z}) & \longrightarrow & H^2_{\mathrm{free}}(M;\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

? Can we also make sense out of $\widehat{H}^k(M;\mathbb{Z})$ for $k \in \mathbb{Z}$?

Differential cohomology

Def: A **differential cohomology theory** is a contravariant functor $\widehat{H}^*(\,\cdot\,;\mathbb{Z}) : \mathsf{Man} \to \mathsf{Ab}^{\mathbb{Z}}$ together with nat. transformations $(\mathrm{curv}, \mathrm{char}, \iota, \kappa)$, such that the following diagram commutes and has exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^{*-1}(M;\mathbb{R})}{H^{*-1}_{\mathrm{free}}(M;\mathbb{Z})} & \longrightarrow & \frac{\Omega^{*-1}(M)}{\Omega^{*-1}_{\mathbb{Z}}(M)} & \xrightarrow{\ \mathrm{d}\ } & \mathrm{d}\Omega^{*-1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow & & \\
0 & \longrightarrow & H^{*-1}(M;\mathbb{T}) & \xrightarrow{\ \kappa\ } & \widehat{H}^*(M;\mathbb{Z}) & \xrightarrow{\ \mathrm{curv}\ } & \Omega^*_{\mathbb{Z}}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{char}} & & \downarrow & & \\
0 & \longrightarrow & H^*_{\mathrm{tor}}(M;\mathbb{Z}) & \longrightarrow & H^*(M;\mathbb{Z}) & \longrightarrow & H^*_{\mathrm{free}}(M;\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Thm: [Simons,Sullivan; Bär,Becker] Differential cohomology theories exist (e.g. Cheeger-Simons theory) and are unique up to a unique natural isomorphism.

Differential cohomology: Examples

◇ Degree $k = 1$: (σ-model with target space $\mathbb{T}$)

- $\widehat{H}^1(M;\mathbb{Z}) \simeq C^\infty(M,\mathbb{T})$
- $\mathrm{curv}(h) = \frac{1}{2\pi i}\mathrm{d}\log h =$ 1-form field strength
- $\mathrm{char}(h) =$ "winding number around the circle" $\in H^1(M;\mathbb{Z})$

◇ Degree $k = 2$: (Maxwell theory)

- $\widehat{H}^2(M;\mathbb{Z}) \simeq \big\{(\text{bundle}, \text{connection})\big\}/\sim$
- $\mathrm{curv}(h) =$ 2-form field strength
- $\mathrm{char}(h) =$ Chern class $\in H^2(M;\mathbb{Z})$

◇ Degree $k = 3$: (Higher Maxwell theory)

- $\widehat{H}^3(M;\mathbb{Z}) \simeq \big\{(\text{gerbe}, \text{connection})\big\}/\sim$
- $\mathrm{curv}(h) =$ 3-form field strength
- $\mathrm{char}(h) =$ Dixmier-Douady class $\in H^3(M;\mathbb{Z})$

◇ Degree $k > 3$: (Higher^{k-2} Maxwell theory) ......

Generalized Maxwell maps and solution subgroups

Maxwell maps and solutions on the category Loc^m

Def: Let δ be the codifferential. The generalized Maxwell map is the natural transformation $\mathrm{MW} := \delta \circ \mathrm{curv} : \widehat{H}^k(\,\cdot\,;\mathbb{Z}) \Rightarrow \Omega^{k-1}(\,\cdot\,)$. The solution subgroups are the kernels $\widehat{\mathfrak{Sol}}^k(M) := \big\{h \in \widehat{H}^k(M;\mathbb{Z}) : \mathrm{MW}(h) = 0\big\}$.

Thm: $\widehat{\mathfrak{Sol}}^k(\,\cdot\,)$ is a subfunctor of $\widehat{H}^k(\,\cdot\,;\mathbb{Z})$ and the following diagram commutes and has exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dfrac{H^{k-1}(M;\mathbb{R})}{H^{k-1}_{\mathrm{free}}(M;\mathbb{Z})} & \longrightarrow & \mathfrak{Sol}^k(M) & \xrightarrow{\ \mathrm{d}\ } & (\mathrm{d}\Omega^{k-1})_\delta(M) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle \iota} & & \downarrow & & \\
0 & \longrightarrow & H^{k-1}(M;\mathbb{T}) & \xrightarrow{\ \kappa\ } & \widehat{\mathfrak{Sol}}^k(M) & \xrightarrow{\ \mathrm{curv}\ } & \Omega^k_{\mathbb{Z},\,\delta}(M) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{char}} & & \downarrow & & \\
0 & \longrightarrow & H^k_{\mathrm{tor}}(M;\mathbb{Z}) & \longrightarrow & H^k(M;\mathbb{Z}) & \longrightarrow & H^k_{\mathrm{free}}(M;\mathbb{Z}) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Thm: $\widehat{\mathfrak{Sol}}^k(f) : \widehat{\mathfrak{Sol}}^k(N) \to \widehat{\mathfrak{Sol}}^k(M)$ is Ab-isomorphism for any Cauchy morphism $f : M \to N$. (Classical time-slice axiom)

Character groups and smooth Pontryagin duals

Sketch of the general idea

- ◇ $\widehat{H}^k(M;\mathbb{Z})$ are configurations, now we want functionals $\widehat{H}^k(M;\mathbb{Z}) \to \mathbb{C}$!
- ◇ As $\widehat{H}^k(M;\mathbb{Z})$ is Abelian group $\exists$ preferred functionals (group characters)

$$\widehat{H}^k(M;\mathbb{Z})^\star := \mathrm{Hom}\big(\widehat{H}^k(M;\mathbb{Z}),\mathbb{T}\big)$$

NB: $\widehat{H}^k(M;\mathbb{Z})^\star$ is **not** a unital $*$-algebra, but an Abelian group. It's clear that $\widehat{H}^k(M;\mathbb{Z})^\star$ can be extended to a unital $*$-algebra by $\mathbb{C}$-linear completion. So let's study this group and construct the algebra later...

??? The character groups are really much too big (we didn't assume any continuity, regularity, etc.). What should we do?

!!! Restrict them by making use of the following observation:

The character group of p-forms $\Omega^p(M)$ has subgroup $\Omega^p_0(M)$ by the identification $\mathcal{W} : \Omega^p_0(M) \to \Omega^p(M)^\star$

$$\mathcal{W}_\varphi(\omega) = \exp\big(2\pi i\langle\varphi,\omega\rangle\big) = \exp\left(2\pi i\int_M \varphi\wedge *\omega\right) .$$

NB: [Harvey,Lawson,Zweck] did a similar thing while working on differential character duality and called such subgroups smooth Pontryagin duals.

List of the smooth Pontryagin duals we need

- Our aim is to dualize (via the exact Hom-functor $\mathrm{Hom}(\,\cdot\,,\mathbb{T})$ and restriction to smooth Pontryagin duals) the fundamental diagram and exact sequences of differential cohomology.

- For everything involving the ordinary cohomology groups we take as smooth Pontryagin duals the full character groups.

- For everything involving differential forms we follow the strategy above and set for the smooth Pontryagin duals:
  - $\big(\frac{\Omega^{k-1}(M)}{\Omega^{k-1}_{\mathbb{Z}}(M)}\big)^\star_\infty := \mathcal{V}^{k-1}(M) := \big\{\varphi\in\Omega^{k-1}_0(M) : \mathcal{W}_\varphi(\omega)=1 \ \forall\omega\in\Omega^{k-1}_{\mathbb{Z}}(M)\big\}$
  - $\big(\mathrm{d}\Omega^{k-1}(M)\big)^\star_\infty := \delta\Omega^k_0(M)$
  - $\Omega^k_{\mathbb{Z}}(M)^\star_\infty := \frac{\Omega^k_0(M)}{\mathcal{V}^k(M)}$
  - $\widehat{H}^k(M;\mathbb{Z})^\star_\infty := \iota^{\star\,-1}\big(\mathcal{V}^{k-1}(M)\big)$, where $\iota^\star := \mathrm{Hom}(\iota,\mathbb{T})$ is dual map between character groups.

Prop: All smooth Pontryagin duals defined above are given by covariant functors from Loc^m to Ab and they separate points of the Abelian groups they act on.

Fundamental theorem for smooth Pontryagin duals

Thm: The following natural diagram commutes and it has exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^k_{\mathrm{free}}(M;\mathbb{Z})^\star & \longrightarrow & H^k(M;\mathbb{Z})^\star & \longrightarrow & H^k_{\mathrm{tor}}(M;\mathbb{Z})^\star & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{char}^\star} & & \downarrow & & \\
0 & \longrightarrow & \dfrac{\Omega^k_0(M)}{\mathcal{V}^k(M)} & \xrightarrow{\mathrm{curv}^\star} & \widehat{H}^k(M;\mathbb{Z})^\star_\infty & \xrightarrow{\kappa^\star} & H^{k-1}(M;\mathbb{T})^\star & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \iota^\star} & & \downarrow & & \\
0 & \longrightarrow & \delta\Omega^k_0(M) & \longrightarrow & \mathcal{V}^{k-1}(M) & \longrightarrow & H^k_{\mathrm{free}}(M;\mathbb{Z})' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Rem: This diagram and exact sequences will later tell us a lot about the classical and quantum field theory, especially their subtheory structure!

Presymplectic structure on smooth Pontryagin duals

Presymplectic structure via Peierls' construction

- Using the generalized Maxwell Lagrangian

$$L(h) = \frac{1}{2}\mathrm{curv}(h) \wedge *\mathrm{curv}(h)$$

and the interpretation of group characters $\mathrm{w} \in \widehat{H}^k(M;\mathbb{Z})^\star_\infty$ as functionals $\mathrm{w} : \widehat{H}^k(M;\mathbb{Z}) \to \mathbb{C}$ via the inclusion $\mathbb{T} \hookrightarrow \mathbb{C}$, we can construct a presymplectic structure on $\widehat{H}^k(M;\mathbb{Z})^\star_\infty$.

- This also requires to define functional derivatives, which can be easily guessed, but also derived from an ∞-dimensional Lie group structure on $\widehat{H}^k(M;\mathbb{Z})$ (the latter is still work in progress). For a tangent vector $[\eta] \in \Omega^{k-1}(M)/\mathrm{d}\Omega^{k-2}(M)$ we set

$$\mathrm{w}^{(1)}(h)\big([\eta]\big) := \lim_{\epsilon \to 0} \frac{\mathrm{w}\big(h + \iota([\epsilon\eta])\big) - \mathrm{w}(h)}{\epsilon} = \mathrm{w}(h)\, 2\pi i\, \langle \iota^\star(\mathrm{w}), [\eta]\rangle \ .$$

- Using Peierls' construction we obtain a Poisson bracket, which is coming from the following presymplectic structure on $\widehat{H}^k(M;\mathbb{Z})^\star_\infty$

$$\widehat{\tau}\big(\mathrm{w},\mathrm{v}\big) = \langle \iota^\star(\mathrm{w}), G\big(\iota^\star(\mathrm{v})\big)\rangle \ .$$

Off-shell classical field theory and its subtheory structure

- Using the presymplectic structure $\widehat{\tau}$ on $\widehat{H}^k(M;\mathbb{Z})^\star_\infty$ and that it is the pull-back via ι^* of a presymplectic structure τ on $\big(\Omega^{k-1}(M)/\Omega^{k-1}_{\mathbb{Z}}(M)\big)^\star_\infty$, we can construct the following covariant functors from Loc^m to PAb:
  - Full classical theory: $\widehat{\mathfrak{G}}^k(\,\cdot\,) := \big(\widehat{H}^k(\,\cdot\,;\mathbb{Z})^\star_\infty, \widehat{\tau}\big)$
  - Topologically trivial classical theory: $\mathfrak{G}^k(\,\cdot\,) := \Big(\big(\Omega^{k-1}(\,\cdot\,)/\Omega^{k-1}_{\mathbb{Z}}(\,\cdot\,)\big)^\star_\infty, \tau\Big)$
  - Classical curvature theory: $\mathfrak{F}^k(\,\cdot\,) := \big(\Omega^k_{\mathbb{Z}}(\,\cdot\,)^\star_\infty, \tau_{\mathfrak{F}}\big)$
  - Classical purely topological theory: $\big(H^k(\,\cdot\,;\mathbb{Z})^\star, 0\big)$

- From the fundamental theorem for smooth Pontryagin duals we get the following diagram in PAb with exact sequences:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathfrak{F}^k(M) & & & & \\
 & & & & \downarrow{\scriptstyle \mathrm{curv}^\star} & & & & \\
0 & \longrightarrow & \overbrace{H^k(M;\mathbb{Z})^\star}^{\text{'magnetic'}} & \xrightarrow{\mathrm{char}^\star} & \widehat{\mathfrak{G}}^k(M) & \xrightarrow{\iota^\star} & \mathfrak{G}^k(M) & \longrightarrow & 0
\end{array}
$$

On-shell theory and even more subtheory structure

- (On-shell theory) := (Off-shell theory)/(vanishing subgroups of solutions)
- Denoting the on-shell functors by the same symbols, we get an even richer subtheory structure:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \underbrace{H^{m-k}(M;\mathbb{R})^\star}_{\text{`electric'}} & \xrightarrow{\ \mathrm{q}^\star\ } & \mathfrak{F}^k(M) & & & & \\
 & & & & \Big\downarrow {\scriptstyle \mathrm{curv}^\star} & & & & \\
0 & \longrightarrow & \underbrace{H^{k}(M;\mathbb{Z})^\star}_{\text{`magnetic'}} & \xrightarrow{\ \mathrm{char}^\star\ } & \widehat{\mathfrak{G}}^k(M) & \xrightarrow{\ \iota^\star\ } & \mathfrak{G}^k(M) & \longrightarrow & 0
\end{array}
$$

- **Important:** The subfunctor $\mathfrak{Charge}^k(\,\cdot\,) := H^{m-k}(\,\cdot\,;\mathbb{R})^\star \oplus H^k(\,\cdot\,;\mathbb{Z})^\star$ describes "electric" and "magnetic" charge observables. It is purely topological and depends only on the homotopy type of spacetime!

Quantization and properties of the QFT

Quantization and properties of the QFT

- Quantization is easily done via the CCR-functor $\mathfrak{CCR} : \mathsf{PAb} \to C^*\mathsf{Alg}$ for presymplectic Abelian groups [Manusceau et al.; Benini,Dappiaggi,Hack,AS]
- **Warning:** $\mathfrak{CCR}$ is **not** an exact functor! Fortunately it preserves monomorphisms, so we have the same subtheory structure!
- Properties of the QFT functor $\widehat{\mathfrak{A}}^k : \mathsf{Loc}^m \to C^*\mathsf{Alg}$:
  - causality axiom ✓
  - time-slice axiom ✓
  - locality axiom $\lightning$ (unless $m = 2$ and $k = 1$)
- **Important:** The violation of the locality axiom can be precisely related to the topological subtheory structure, namely

Thm: For any Loc^m-morphism $f : M \to N$ the $C^*\mathsf{Alg}$-morphism $\widehat{\mathfrak{A}}^k(f) : \widehat{\mathfrak{A}}^k(M) \to \widehat{\mathfrak{A}}^k(N)$ is injective **if and only if** the Ab-morphism $\mathfrak{Charge}^k(f) : \mathfrak{Charge}^k(M) \to \mathfrak{Charge}^k(N)$ is injective.

- In easy words: The purely topological subtheory is the **only** source of violations of the locality axiom!
- Or even easier: "Magnetic" and "electric" charges are the only things that can screw up locality in Abelian gauge theories of any degree!

Conclusion

Conclusions

- Differential cohomology is a very effective technique to construct Abelian gauge theories in any degree (i.e. $k-2$-gerbes with connections).
- From the fundamental diagram and exact sequences defining a differential cohomology theory (up to nat. iso.) already a lot of properties of the classical and quantum field theory follow, e.g. the existence of subfunctors.
- Most interesting is the subfunctor $\mathfrak{Charge}^k(\,\cdot\,) := H^{m-k}(\,\cdot\,;\mathbb{R})^\star \oplus H^k(\,\cdot\,;\mathbb{Z})^\star$, which depends only on the topology of spacetime.
- It is fair to call $\mathfrak{CCR}(\mathfrak{Charge}^k(\,\cdot\,))$ a topological QFT (from the perspective of an algebraic quantum field theorist, not from the perspective of Atiyah).
- So Abelian quantum gauge theories have topological sub-QFTs!
- The infamous violation of the locality axiom is precisely due to this topological sub-QFT.
- **Open problems/Work in progress:** What is the role of the group of flat characters? "θ-angle" representations? Abelian S-duality? Differential K-theory?