Supergeometry in locally covariant QFT

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Motivation

◇ Supersymmetric QFTs are interesting because of their unexpected renormalization properties

$\rightsquigarrow$ 'non-renormalization theorems' [Grisaru,Rocek,Siegel; Seiberg; ...]

◇ Usual framework is very restrictive: super-QFTs on super-Minkowski space.

Q: Do the non-renormalization theorems survive on curved supermanifolds?

A: We don't know yet! Analyzing this question requires heavy machinery such as perturbative locally covariant QFT [Brunetti,Fredenhagen; ...].

◇ Our goal is to develop mathematical techniques which will eventually allow us to give an answer to this question.

◇ **Goals of my talk:**

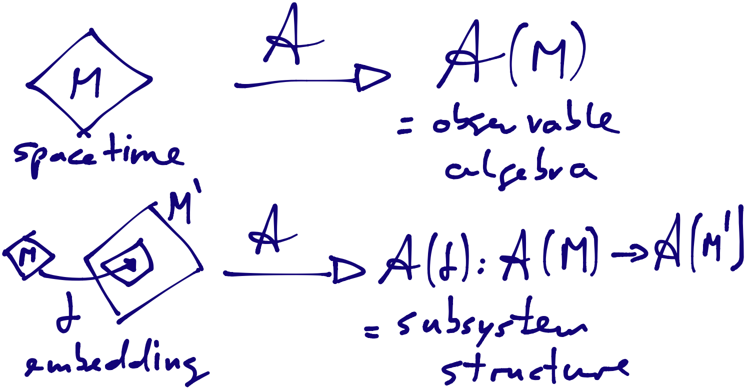
1. Understand the structure of supersymmetry transformations in locally covariant QFT. (This will be an essential ingredient for proving non-renormalization theorems!)
2. Construct non-interacting models of super-QFTs and analyze their structure. (Perturbatively interacting models then will be deformations of those.)

Outline

1. What is locally covariant QFT?
2. Super-Cartan supermanifolds
3. A too naive approach to super-QFTs
4. Enriched category theory and super-QFTs
5. Examples in $1|1$ and $3|2$-dimensions
6. Summary and outlook

What is locally covariant QFT?

What should a QFT do?

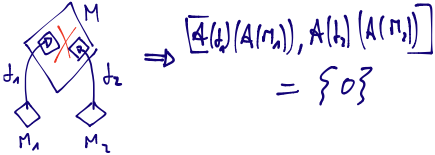


+ suitable axioms

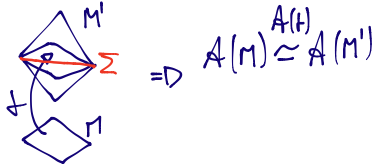
Locally covariant QFT

Def: A locally covariant QFT (LCQFT) [Brunetti,Fredenhagen,Verch] is a functor $\mathfrak{A} : \mathsf{Loc} \to (C)^*\mathsf{Alg}$, such that

(i) if $f_1 : M_1 \to M$ and $f_2 : M_2 \to M$ are causally disjoint, then $\mathfrak{A}(f_1)[\mathfrak{A}(M_1)]$ and $\mathfrak{A}(f_2)[\mathfrak{A}(M_2)]$ commute as subalgebras of $\mathfrak{A}(M)$ *(causality axiom)*



(ii) if $f : M \to M'$ is Cauchy morphism (i.e. $f[M] \subseteq M'$ contains Cauchy surface), then $\mathfrak{A}(f)$ is isomorphism *(time-slice axiom)*



(iii) all $\mathfrak{A}(f) : \mathfrak{A}(M) \to \mathfrak{A}(M')$ are injective *(locality axiom)*

Super-Cartan supermanifolds

Supermanifolds

◇ Recall that there are two equivalent ways to look at smooth manifolds:

1. A topological space $\widetilde{M}$ which locally looks like $\mathbb{R}^n$ (i.e. $\exists$ smooth atlas).
2. A sheaf $M = (\widetilde{M}, \mathscr{O}_M)$ of commutative algebras on $\widetilde{M}$ which locally looks like $(\mathbb{R}^n, C^\infty_{\mathbb{R}^n})$.

Def: A supermanifold is a sheaf $M = (\widetilde{M}, \mathscr{O}_M)$ of supercommutative superalgebras which locally looks like $\mathbb{R}^{n|m} := (\mathbb{R}^n, C^\infty_{\mathbb{R}^n} \otimes \bigwedge^\bullet \mathbb{R}^m)$.

◇ Most differential geometric concepts easily generalize to supermanifolds:

- super-vector fields: sheaf of superderivations Der_M
- super-one-forms: dual sheaf $\Omega^1_M = \underline{\mathrm{Hom}}_{\mathscr{O}_M}(\mathrm{Der}_M, \mathscr{O}_M)$
- super-forms: $\Omega^\bullet_M = \bigwedge^\bullet \Omega^1_M$ with differential $\mathrm{d} : \Omega^\bullet_M \to \Omega^{\bullet+1}_M$

NB: If $m \neq 0$, there are no top-degree forms $\Rightarrow$ super-integration differs!

◇ *Solution:* Berezin integration $\int_M : \mathrm{Ber}_\mathrm{c}(\Omega^1_M) \to \mathbb{R}$ (for M oriented) defined by globalizing $\int_{\mathbb{R}^{n|m}} [\mathrm{d}x^1, \dots, \mathrm{d}x^n, \mathrm{d}\theta^1, \dots, \mathrm{d}\theta^m]\, f = \int_{\mathbb{R}^n} \mathrm{d}^n x\, f_{(1,\dots,1)}$, where we used odd-coordinate expansion $f = f^{\text{lower order}} + \theta^m \cdots \theta^1\, f_{(1,\dots,1)}$.

Super-Cartan structures

- **Wanted:** 'Superspacetime' structure on supermanifolds.
- Motivated by the superspace formulation of supergravity [Wess,Zumino; ...], super-Cartan structures are the correct concept:

Def: A (globally trivial and trivialized) super-Cartan structure on M is a pair (Ω, E) consisting of an even super-spin connection $\Omega \in \Omega^1(M, \mathfrak{spin})$ and a non-degenerate and even supervielbein $E \in \Omega^1(M, \mathfrak{t})$.

We call the triple $\boldsymbol{M} = (M, \Omega, E)$ a super-Cartan supermanifold.

NB: For defining super-Cartan structures we need a supertranslation super-Lie algebra $\mathfrak{t}$ and a super-Poincaré super-Lie algebra $\mathfrak{sp}$, specified by the super-Lie algebra extension $\mathfrak{t} \longrightarrow \mathfrak{sp} \longrightarrow \mathfrak{spin}$.

These structures can be derived from the following data:

- real vector space W,
- Lorentz metric $g : W \otimes W \to \mathbb{R}$,
- $\mathrm{Spin}(W, g)$-representation S and equivariant symmetric pairing $\Gamma : S \otimes S \to W$.

For later use, we also want a positive time-like cone $C \subset W$ (time-orientation), an equivariant metric or symplectic structure $\epsilon : S \otimes S \to \mathbb{R}$ and orientations o_W and o_S.

Interpretation: The representation theoretic data $(W, g, S, \Gamma, C, \epsilon, o_W, o_S)$ is the local (oriented and time-oriented) model space for the super-Cartan supermanifolds.

Globally hyperbolic super-Cartan supermanifolds

Prop: There is a functor $\widetilde{\cdot}$: SCart $\to$ otLor.

To any super-Cartan supermanifold $\boldsymbol{M} = (M, \Omega, E)$ it assigns the oriented and time-oriented Lorentz manifold $\widetilde{\boldsymbol{M}}$ with underlying manifold $(\widetilde{M}, \mathscr{O}_M/J_M)$ (J_M is sheaf of nilpotents) and metric, orientation and time-orientation determined by the pull-back $\widetilde{E} = \iota^*_{\widetilde{M},M}(E)$ of the supervielbein along the canonical embedding $\iota_{\widetilde{M},M} : \widetilde{M} \to M$.

Def:

(i) The causal future/past of a subset $A \subseteq \widetilde{M}$ in a super-Cartan supermanifold $\boldsymbol{M}$ is defined in terms of its underlying oriented and time-oriented Lorentz manifold $\widetilde{\boldsymbol{M}}$, i.e. $J^{\pm}_{\boldsymbol{M}}(A) := J^{\pm}_{\widetilde{\boldsymbol{M}}}(A) \subseteq \widetilde{M}$.

(ii) A super-Cartan supermanifold $\boldsymbol{M}$ is called globally hyperbolic if $\widetilde{\boldsymbol{M}}$ is globally hyperbolic.

(iii) The category ghSCart has as objects all globally hyperbolic super-Cartan supermanifolds and as morphisms all SMan-morphisms $\chi : M \to M'$ such that

1. $\widetilde{\chi} : \widetilde{M} \to \widetilde{M'}$ is open and causally compatible,
2. $\chi : M \to M'|_{\widetilde{\chi}(\widetilde{M})}$ is SMan-isomorphism,
3. $\chi^*(\Omega') = \Omega$ and $\chi^*(E') = E$.

NB: One could weaken 3. by demanding the identities ‘up to local $\mathrm{Spin}(W, g)$-transformations’. Requires understanding of super-principal bundles and their automorphisms $\to$ future work!

A too naive approach to super-QFTs

Axiomatic definition of super-field theories

◇ **Goal:** Give a simple characterization of classical super-field theories, which allows us to give a general construction of super-QFT functors.

Def: A super-field theory is specified by the following data:

1. A choice of representation theoretic data $(W, g, S, \Gamma, C, \epsilon, o_W, o_S)$.
2. A full subcategory SLoc of ghSCart.
3. A natural transformation $P : \mathscr{O} \Rightarrow \mathscr{O}$ of functors $\mathscr{O}$: SLoc$^{\mathrm{op}} \to$ SVec, such that any $P_{\boldsymbol{M}}$ is a formally super-self adjoint and super-Green's hyperbolic super-differential operator (with $|P_{\boldsymbol{M}}| = \dim(S)$ mod 2).

Rem:

1. This data fixes the local model space and hence the dimension of spacetime and the amount of supersymmetry.
2. In SLoc we collect all admissible super-Cartan supermanifolds, e.g. those satisfying the supergravity supertorsion constraints.
3. The natural super-differential operator $P_{\boldsymbol{M}}$ governs the dynamics of the superfield $\Phi \in \mathscr{O}(\boldsymbol{M})$, i.e. the equation of motion is $P_{\boldsymbol{M}}(\Phi) = 0$.

   For technical reasons we want the $P_{\boldsymbol{M}}$ to be formally super-self adjoint w.r.t. the natural pairing $\langle F_1, F_2 \rangle_{\boldsymbol{M}} = \int_M \mathrm{Ber}(E)\, F_1\, F_2$ (Berezin integral) and to admit retarded/advanced super-Green's operators $G^{\pm}_{\boldsymbol{M}} : \mathscr{O}_{\mathrm{c}}(\boldsymbol{M}) \to \mathscr{O}(\boldsymbol{M})$.

Construction of super-QFTs

Theorem (Existence of super-QFT functors)

Given any super-field theory according to the definition above, there exists a functor $\mathfrak{A} : \mathsf{SLoc} \to \mathsf{S^*Alg}$ *satisfying*

- *Locality: For any* SLoc*-morphism* $\chi : \boldsymbol{M} \to \boldsymbol{M}'$*, the* S*Alg*-morphism* $\mathfrak{A}(\chi) : \mathfrak{A}(\boldsymbol{M}) \to \mathfrak{A}(\boldsymbol{M}')$ *is monic.*
- *Causality: Given any two* SLoc*-morphisms* $\boldsymbol{M}_1 \xrightarrow{\chi_1} \boldsymbol{M} \xleftarrow{\chi_2} \boldsymbol{M}_2$ *such that the images of the reduced* otLor*-morphisms* $\widetilde{\boldsymbol{M}_1} \xrightarrow{\widetilde{\chi_1}} \widetilde{\boldsymbol{M}} \xleftarrow{\widetilde{\chi_2}} \widetilde{\boldsymbol{M}_2}$ *are causally disjoint, then*

$$a_1\, a_2 + (-1)^{|P_{\boldsymbol{M}}|+1}\, (-1)^{|a_1|\,|a_2|}\, a_2\, a_1 = 0\ ,$$

  for all homogeneous $a_1 \in \mathfrak{A}(\chi_1)(\mathfrak{A}(\boldsymbol{M}_1))$ *and* $a_2 \in \mathfrak{A}(\chi_2)(\mathfrak{A}(\boldsymbol{M}_2))$*.*
- *Time-slice axiom: Given any Cauchy* SLoc*-morphism* $\chi : \boldsymbol{M} \to \boldsymbol{M}'$*, then* $\mathfrak{A}(\chi) : \mathfrak{A}(\boldsymbol{M}) \to \mathfrak{A}(\boldsymbol{M}')$ *is a* S*Alg*-isomorphism.*

The proof is a quite simple generalization of the corresponding proof for ordinary QFTs, see e.g. [Bär,Ginoux,Pfäffle; . . .].

Properties of the super-QFT

Prop: The functor $\mathfrak{A} : \mathsf{SLoc} \to \mathsf{S^*Alg}$ has a locally covariant quantum field of type $\mathscr{O}_{\mathrm{c}} : \mathsf{SLoc} \to \mathsf{SVec}$ (i.e. a scalar super-quantum field)

- Explicitly, there exists a natural transformation $\Phi : \mathscr{O}_{\mathrm{c}} \Rightarrow \mathfrak{A}$ such that

$$\Phi_{\boldsymbol{M}}(P_{\boldsymbol{M}}(F)) = 0 \ ,$$

$$\Phi_{\boldsymbol{M}}(F_1)\Phi_{\boldsymbol{M}}(F_2) + (-1)^{|P_{\boldsymbol{M}}|+1}\,(-1)^{|F_1|\,|F_2|}\,\Phi_{\boldsymbol{M}}(F_2)\Phi_{\boldsymbol{M}}(F_1) = \beta\,\langle G_{\boldsymbol{M}}(F_1), F_2\rangle_{\boldsymbol{M}} \ ,$$

  i.e. Φ satisfies the equation of motion $P_{\boldsymbol{M}}(\Phi_M) = 0$ and the SCCR/SCAR-properties ($\beta = i$ if $P_{\boldsymbol{M}}$ is even and $\beta = 1$ if $P_{\boldsymbol{M}}$ is odd).

- **Problem:** We can decompose $\mathscr{O}_{\mathrm{c}} = \mathscr{O}_{\mathrm{c}}^{\mathrm{even}} \oplus \mathscr{O}_{\mathrm{c}}^{\mathrm{odd}}$ and obtain that the restrictions $\Phi^{\mathrm{even/odd}} : \mathscr{O}_{\mathrm{c}}^{\mathrm{even/odd}} \Rightarrow \mathfrak{A}$ are also natural transformations.
  - $\rightsquigarrow$ bosonic/fermionic component quantum-fields are also natural
  - $\rightsquigarrow$ supersymmetry transformations not correctly implemented ☹

- **Reason:** The categories SLoc, <u>SVec</u> and S*Alg do not contain enough morphisms!

Enriched category theory and super-QFTs

Basic idea: Morphism supersets

- The SLoc-morphisms $\chi : \boldsymbol{M} \to \boldsymbol{M}'$ are all even, so no supersymmetry transformations possible. ($\chi^* : \mathscr{O}(\boldsymbol{M}') \to \mathscr{O}(\boldsymbol{M})$ always preserves even/odd-splitting.)
- **Idea:** "Fatten up" M and M' by superpoints $\mathrm{pt}_n = (\{\star\}, \Lambda_n)$ and consider morphisms $\chi : \mathrm{pt}_n \times M \to \mathrm{pt}_n \times M'$ preserving the fibrations over pt_n.
- $\chi^* : \Lambda_n \otimes \mathscr{O}(M') \to \Lambda_n \otimes \mathscr{O}(M)$ is Λ_n-linear superalgebra morphism which schematically acts on $\mathbb{1} \otimes F \in \Lambda_n \otimes \mathscr{O}(M')$ as

$$\chi^*(\mathbb{1} \otimes F) = \sum_I \xi^I \otimes \chi_I^*(F) \quad , \qquad \text{with } \xi^I \text{ basis of } \Lambda_n \ .$$

$\Rightarrow$ If $\chi_I^* \neq 0$ for some odd ξ^I, then χ^* does **not** preserve splitting $\Lambda_n \otimes \mathscr{O}^{\text{even/odd}}(M^{(\prime)})$ $\Rightarrow$ SUSY transformations!

- Fixing any superpoint pt_n is not clever (which one should we fix?) $\Rightarrow$ we have to work functorially over **all** superpoints!

Def: The monoidal category of supersets is defined by $\mathsf{SSet} := \mathrm{Fun}(\mathsf{SPt}^{\mathrm{op}}, \mathsf{Set})$.

Ex: "Fattened up" morphisms $\mathsf{FatMorph}(M, M')$ are a superset; explicitly, $\mathsf{FatMorph}(M, M')(\mathrm{pt}_n) := \mathrm{Hom}_{\mathsf{SMan}/\mathrm{pt}_n}(\mathrm{pt}_n \times M, \mathrm{pt}_n \times M')$.

Enriched categories

- ◇ In a usual category, the morphisms between two objects have to form a set. We however want to have morphism supersets between objects in SLoc.

- ◇ How can we do that? 

Def: Let $(\mathsf{V}, \otimes, I)$ be (strict) monoidal category. A V-category C (or V-enriched category) consists of

- a class of objects $\mathrm{Ob}(\mathsf{C})$,
- an 'object of morphisms' $\mathsf{C}(A, B)$ in V, for all $A, B \in \mathrm{Ob}(\mathsf{C})$,
- a 'composition morphism' $\bullet_{A,B,C} : \mathsf{C}(B, C) \otimes \mathsf{C}(A, B) \to \mathsf{C}(A, C)$ in V, for all $A, B, C \in \mathrm{Ob}(\mathsf{C})$,
- an 'identity on A morphism' $\mathbf{1}_A : I \to \mathsf{C}(A, A)$ in V, for all $A \in \mathrm{Ob}(\mathsf{C})$,

such that the $\bullet_{\_,\_,\_}$ and $\mathbf{1}_{\_}$ satisfy associativity and unity conditions.

Rem: Ordinary categories are Set-categories.

Rem: Enriched functors and natural transformations are easy to define. (Homework!)

Associativity and unity conditions

$$
\begin{array}{ccc}
\mathsf{C}(C,D)\otimes\mathsf{C}(B,C)\otimes\mathsf{C}(A,B) & \xrightarrow{\mathrm{id}_{\mathsf{C}(C,D)}\otimes\bullet_{A,B,C}} & \mathsf{C}(C,D)\otimes\mathsf{C}(A,C)\\
{\scriptstyle \bullet_{B,C,D}\otimes\mathrm{id}_{\mathsf{C}(A,B)}}\big\downarrow & & \big\downarrow{\scriptstyle \bullet_{A,C,D}}\\
\mathsf{C}(B,D)\otimes\mathsf{C}(A,B) & \xrightarrow[\bullet_{A,B,D}]{} & \mathsf{C}(A,D)
\end{array}
$$

$$
\begin{array}{ccccc}
I\otimes\mathsf{C}(A,B) & & & & \mathsf{C}(A,B)\otimes I\\
{\scriptstyle \mathbf{1}_B\otimes\mathrm{id}_{\mathsf{C}(A,B)}}\big\downarrow & \searrow^{\simeq} & & \swarrow^{\simeq} & \big\downarrow{\scriptstyle \mathrm{id}_{\mathsf{C}(A,B)}\otimes\mathbf{1}_A}\\
\mathsf{C}(B,B)\otimes\mathsf{C}(A,B) & \xrightarrow[\bullet_{A,B,B}]{} & \mathsf{C}(A,B) & \xleftarrow[\bullet_{A,A,B}]{} & \mathsf{C}(A,B)\otimes\mathsf{C}(A,A)
\end{array}
$$

Axioms for enriched super-field theories

- By "fattening up" the morphisms in SLoc, <u>SVec</u> and S*Alg, we get SSet-categories eSLoc, <u>eSVec</u> and eS*Alg. (Quite technical!)

- **Important:** Enriching does change the morphisms, but not the objects!!!

- The global section functor $\mathscr{O} : \mathsf{SLoc}^{\mathrm{op}} \to \underline{\mathsf{SVec}}$ extends to a SSet-functor $\mathfrak{e}\mathscr{O} : \mathsf{eSLoc}^{\mathrm{op}} \to \underline{\mathsf{eSVec}}$.

Def: An enriched super-field theory is a super-field theory, such that the natural transformation $P : \mathscr{O} \Rightarrow \mathscr{O}$ lifts to a SSet-natural trafo $eP : \mathfrak{e}\mathscr{O} \Rightarrow \mathfrak{e}\mathscr{O}$.

Prop: A super-field theory is an enriched super-field theory if and only if the diagram

$$\begin{array}{ccc} \Lambda_n \otimes \mathscr{O}(\boldsymbol{M}') & \xrightarrow{\ \mathrm{id}_{\Lambda_n} \otimes P_{\boldsymbol{M}'}\ } & \Lambda_n \otimes \mathscr{O}(\boldsymbol{M}') \\ \Big\downarrow \chi^* & & \Big\downarrow \chi^* \\ \Lambda_n \otimes \mathscr{O}(\boldsymbol{M}) & \xrightarrow[\ \mathrm{id}_{\Lambda_n} \otimes P_{\boldsymbol{M}}\ ]{} & \Lambda_n \otimes \mathscr{O}(\boldsymbol{M}) \end{array}$$

commutes for all fattened morphisms $\chi \in \mathsf{eSLoc}(\boldsymbol{M}, \boldsymbol{M}')(\mathrm{pt}_n)$.

- *Physically:* SUSY covariance of the super-differential operators $P_{\boldsymbol{M}}$.

Construction of enriched super-QFTs

Theorem (Existence of enriched super-QFT functors)

Given any enriched super-field theory according to the definition above, there exists a SSet*-functor* $\mathfrak{eA} : \mathsf{eSLoc} \to \mathsf{eS^*Alg}$ *satisfying*

- *Enriched locality: For any* $\chi \in \mathsf{eSLoc}(\boldsymbol{M}, \boldsymbol{M}')(\mathrm{pt}_n)$*, we have that* $(\mathfrak{eA}_{\boldsymbol{M},\boldsymbol{M}'})_{\mathrm{pt}_n}(\chi) \in \mathsf{eS^*Alg}(\mathfrak{A}(\boldsymbol{M}), \mathfrak{A}(\boldsymbol{M}'))(\mathrm{pt}_n)$ *is monic.*
- *Enriched causality: Given any* $\chi_1 \in \mathsf{eSLoc}(\boldsymbol{M}_1, \boldsymbol{M})(\mathrm{pt}_n)$ *and* $\chi_2 \in \mathsf{eSLoc}(\boldsymbol{M}_2, \boldsymbol{M})(\mathrm{pt}_n)$*, such that the images of the reduced* otLor*-morphisms* $\widetilde{\boldsymbol{M}_1} \xrightarrow{\widetilde{\chi_1}} \widetilde{\boldsymbol{M}} \xleftarrow{\widetilde{\chi_2}} \widetilde{\boldsymbol{M}_2}$ *are causally disjoint, then*

$$A_1\, A_2 + (-1)^{|P_{\boldsymbol{M}}|+1}\, (-1)^{|A_1|\,|A_2|}\, A_2\, A_1 = 0 \ ,$$

  for all homogeneous $A_1 \in (\mathfrak{eA}_{\boldsymbol{M}_1,\boldsymbol{M}})_{\mathrm{pt}_n}(\chi_1)\big(\Lambda_n^{\mathbb{C}} \otimes_{\mathbb{C}} \mathfrak{A}(\boldsymbol{M}_1)\big)$ *and* $A_2 \in (\mathfrak{eA}_{\boldsymbol{M}_2,\boldsymbol{M}})_{\mathrm{pt}_n}(\chi_2)\big(\Lambda_n^{\mathbb{C}} \otimes_{\mathbb{C}} \mathfrak{A}(\boldsymbol{M}_2)\big)$*.*
- *Enriched time-slice axiom: Given any* $\chi \in \mathsf{eSLoc}(\boldsymbol{M}, \boldsymbol{M}')(\mathrm{pt}_n)$ *such that* $\widetilde{\chi} : \widetilde{\boldsymbol{M}} \to \widetilde{\boldsymbol{M}'}$ *is Cauchy, then* $(\mathfrak{eA}_{\boldsymbol{M},\boldsymbol{M}'})_{\mathrm{pt}_n}(\chi) \in \mathsf{eS^*Alg}(\mathfrak{A}(\boldsymbol{M}), \mathfrak{A}(\boldsymbol{M}'))(\mathrm{pt}_n)$ *is an isomorphism.*

Properties of the enriched super-QFT

Prop: The SSet-functor $\mathfrak{eA} : \mathsf{eSLoc} \to \mathsf{eS^*Alg}$ has an enriched locally covariant quantum field of type $\mathfrak{e}\mathscr{O}_{\mathrm{c}} : \mathsf{eSLoc} \to \mathsf{eSVec}$.

- Explicitly, there exists a SSet-natural transformation $\Phi : \mathfrak{e}\mathscr{O}_{\mathrm{c}} \Rightarrow \mathfrak{eA}$ with components $\Phi_{\boldsymbol{M}}$ ($=$ natural transformations of functors from $\mathsf{SPt}^{\mathrm{op}}$ to SSet) satisfying the equation of motion and SCCR/SCAR-properties.

- **Important:** By construction, $\Phi_{\boldsymbol{M}}$ transforms also good under 'fattened up' morphisms, i.e. SUSY transformations. As a consequence, it does **not** restrict to $\mathscr{O}_{\mathrm{c}}^{\mathrm{even/odd}}$ provided there exists a 'fattened up' $\chi : \mathrm{pt}_n \times \boldsymbol{M} \to \mathrm{pt}_n \times \boldsymbol{M}'$, such that $\chi^*(\mathbb{1} \otimes F) = \sum_I \xi^I \otimes \chi_I^*(F)$ with $\chi_I^* \neq 0$ for some odd ξ^I.

- *Physical relevance:* Enriched natural transformations are more restrictive; in particular one can not treat bosonic and fermionic components independently. This will reduce the renormalization freedom when constructing super-Wick-polynomials in terms of SSet-natural transformations!

  (Work in progress; master's thesis of Angelo Cuzzola.)

Examples in $1|1$ and $3|2$-dimensions

The superparticle in $1|1$-dimensions

- Let's start with the lowest-dimensional example:
  - *Representation theoretic data:* Choose $W = \mathbb{R}$ with standard metric g. Then $\mathrm{Spin}(W,g) \simeq \{\pm 1\}$ and take $S = \mathbb{R}$ spin-representation with $\Gamma(s_1, s_2) = s_1 s_2$. In orthonormal basis, the super-Poincaré super-Lie algebra is generated by $p \in W$ and $q \in S$, satisfying $[p,p] = 0$, $[p,q] = 0$ and $[q,q] = -2p$.
  - *Category* SLoc*:* Take super-Cartan supermanifolds $\boldsymbol{M} = (M, \Omega = 0, E)$ with underlying $\widetilde{M} \subseteq \mathbb{R}$. Demand that $\boldsymbol{M}$ satisfies supertorsion constraints $\Rightarrow$ $E = (\mathrm{d}t + \theta\,\mathrm{d}\theta) \otimes p + \mathrm{d}\theta \otimes q$.
  - *SDiffOp:* Take dual superderivations $X = \partial_t$ and $D = \partial_\theta + \theta\partial_t$ and define $P_{\boldsymbol{M}} := X \circ D = \partial_t\partial_\theta + \theta\partial_t^2$. Then P is natural, formally super self-adjoint and super-Green's hyperbolic. Component equations: $P_{\boldsymbol{M}}(\Phi) = \partial_t\psi + \theta\,\partial_t^2\phi = 0$

Prop: This data specifies an enriched super-field theory.

- The enriched automorphism group $\mathrm{eSLoc}(\boldsymbol{M}, \boldsymbol{M})$ is a group object in SSet, which is nontrivial for any $\boldsymbol{M}$ and representable by $\mathbb{R}^{1|1}$ if $\widetilde{M} = \mathbb{R}$.
- Expanding $\Phi_{\boldsymbol{M}}(f + \theta h) = \psi_{\boldsymbol{M}}(f) + \phi_{\boldsymbol{M}}(h)$, one recovers from the SSet-functor $\mathfrak{eA} : \mathrm{eSLoc} \to \mathrm{eS}^*\mathrm{Alg}$ the ordinary SUSY transformations $\delta^{\mathrm{SUSY}}\phi_{\boldsymbol{M}}(h) = -\psi_{\boldsymbol{M}}(h)$ and $\delta^{\mathrm{SUSY}}\psi_{\boldsymbol{M}}(f) = \phi_{\boldsymbol{M}}(\partial_t f)$.

The Wess-Zumino model in $3|2$-dimensions

- In $3|2$-dimensions things get much more complicated/lengthy. A rough sketch of the construction of the free Wess-Zumino model is as follows:
  - *Representation theoretic data:* Can be looked up in textbooks...
  - *Category* SLoc*:* For simplicity assume that $\boldsymbol{M}$ has trivial Batchelor bundle, $\mathscr{O}_M \simeq C^\infty_{\widetilde{M}} \otimes \bigwedge^\bullet \mathbb{R}^2$. Demand supertorsion constraints to simplify form of supervielbeins $E = e^\alpha \otimes p_\alpha + \zeta^a \otimes q_a$. (Also needed for s.f.s.a.!)
  - *SDiffOp:* Using the dual superderivations X_α and D_a and the covariant differential $\mathrm{d}_\Omega = \mathrm{d} + \Omega$, we define super-differential operator

$$P_{\boldsymbol{M}}(\Phi) = \frac{1}{2}\, \epsilon^{ab}\, \langle D_b, \mathrm{d}_\Omega(D_a \Phi)\rangle + m\, \Phi$$

Prop: This data specifies an enriched super-field theory.

Rem: It is astonishing that the supertorsion constraints restrict $\boldsymbol{M}$ such that the only d.o.f. are given by an ordinary vielbein $\widetilde{E}$ on $\widetilde{M}$. The equation of motion $P_{\boldsymbol{M}}(\Phi) = 0$ implies the component equations, for $\Phi = \phi + \theta^a \psi_a + \frac{\theta^2}{2}\eta$,

$$-\eta + m\,\varphi = 0\ , \quad i\nabla\!\!\!\!/\,\psi_a + m\,\psi_a = 0\ , \quad \Box\varphi + m\,\eta = 0\ .$$

Summary and outlook

Summary and outlook

◇ We have:

- understood that enriched category theory plays a major role in combining supergeometry with locally covariant QFT. (SUSY transformations!)
- a general construction of enriched non-interacting super-QFT functors $\mathfrak{eA} : \mathsf{eSLoc} \to \mathsf{eS^*Alg}$ starting from "simple" representation theoretic and geometric data.
- found that the superparticle and the Wess-Zumino model in $3|2$-dimensions fits into our framework.

◇ We still have to:

- understand how to use the powerful framework of enriched natural transformations to construct super-Wick-polynomials and study their renormalization properties.
- understand how supergeometry improves the construction of perturbatively interacting QFTs. (Need super-microlocal analysis!!)
- include global aspects of super-Cartan geometry and in particular super-gauge invariance.