Quantized Abelian principal connections on Lorentzian manifolds

Alexander Schenkel

Department of Mathematics, University of Wuppertal

Seminar @ the Differential Geometry Group, University of Potsdam

April 15, 2013

Based on joint work with Marco Benini and Claudio Dappiaggi:

(i) [arXiv:1303.2515 [math-ph]]

(ii) [arXiv:1210.3457 [math-ph]] to appear in Annales Henri Poincaré

and on ongoing work with Benini, Dappiaggi, Gottschalk and Hack.

Motivation

Motivation

- Part of the most important equations in physics are Maxwell's equations

$$\mathrm{div}\mathbf{E} = 0 \ , \quad \mathrm{div}\mathbf{B} = 0 \ , \quad \mathrm{rot}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \ , \quad \mathrm{rot}\mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} \ .$$

- They describe a huge variety of physical phenomena, e.g. electrostatics, magnetostatics, EM waves, light, ... (on Minkowski spacetime $\mathbb{R}^{1,3}$)
- Maxwell's equations can be generalized to oriented Lorentzian manifolds

$$\delta\mathcal{F} = 0 \ , \quad \mathrm{d}\mathcal{F} = 0 \ , \quad \mathcal{F} \in \Omega^2(M) \quad \text{(field strength tensor)}$$

- Physics tells us that $\mathcal{F}$ is **not** the fundamental degree of freedom of electromagnetism, but it is derived from a "gauge field".
- The gauge field $\omega \in \mathrm{Con}(P)$ is a connection form on a principal $G = U(1)$-bundle $P \xrightarrow{\pi} M$, $\mathcal{F}_\omega := \mathrm{d}\omega \in \Omega^2_{\mathrm{hor}}(P,\mathfrak{g})^{\mathrm{eqv}} \simeq \Omega^2(M,\mathfrak{g})$ is the curvature and Maxwell's equations read

$$\delta\mathcal{F}_\omega = 0$$

- **Goal of my talk:** Quantization of connections and Maxwell's equations using algebraic quantum field theory techniques.

Outline

1. Locally covariant QFT: Category theory meets QFT

2. Quantized Abelian principal connections

3. Topological quantum fields in AQFT

4. Charge-zero functor is a locally covariant QFT

5. Towards a very good functor for the Maxwell field

6. Conclusions and outlook

Locally covariant QFT: Category theory meets QFT

The essentials of locally covariant QFT

- The basic idea of locally covariant QFT [à la Brunetti, Fredenhagen, Verch] is to associate to each spacetime M an algebra of quantum field observables $\mathfrak{A}(M)$, such that this association "behaves well" under embeddings of spacetimes $f : M_1 \to M_2$.
- This can be made precise by using tools from category theory. Let us define the categories:

Man: $\mathrm{Obj}(\mathsf{Man})$ are oriented, time-oriented and globally hyperbolic Lorentzian manifolds.

$\mathrm{Mor}(\mathsf{Man})$ are orientation and time-orientation preserving isometric embeddings, such that the image is causally compatible and open.

*Alg: $\mathrm{Obj}(^*\mathsf{Alg})$ are unital $*$-algebras over $\mathbb{C}$.

$\mathrm{Mor}(^*\mathsf{Alg})$ are **injective** unital $*$-algebra homomorphisms.

Def: A locally covariant QFT is a covariant functor $\mathfrak{A} : \mathsf{Man} \to {}^*\mathsf{Alg}$, satisfying

1. the causality property,
2. the time-slice axiom.

$\to$ See the blackboard for some explaining pictures!

Example: The real Klein-Gordon field

- To any object M in Man we associate the Klein-Gordon operator

$$P : C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R}) \ , \quad \phi \mapsto P(\phi) = \Box(\phi) - m^2\phi \ .$$

- The retarded/advanced Green's operators $G^\pm : C^\infty_0(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ are unique, and the causal propagator $G := G^+ - G^- : C^\infty_0(M,\mathbb{R}) \to C^\infty_{\mathrm{sc}}(M,\mathbb{R})$ generates all solutions of $P(\phi) = 0$ due to the following exact complex

$$\{0\} \longrightarrow C^\infty_0(M,\mathbb{R}) \xrightarrow{\ P\ } C^\infty_0(M,\mathbb{R}) \xrightarrow{\ G\ } C^\infty_{\mathrm{sc}}(M,\mathbb{R}) \xrightarrow{\ P\ } C^\infty_{\mathrm{sc}}(M,\mathbb{R})$$

- The vector space $\mathcal{E} := C^\infty_0(M,\mathbb{R})/P[C^\infty_0(M,\mathbb{R})]$ can be equipped with the symplectic structure

$$\tau : \mathcal{E}\times\mathcal{E} \to \mathbb{R} \ , \quad \big([\varphi],[\psi]\big) \mapsto \tau\big([\varphi],[\psi]\big) = \int_M \mathrm{vol}\ \varphi\, G(\psi)$$

NB: We interpret $\varphi \in C^\infty_0(M,\mathbb{R})$ as a label for functionals

$$\mathcal{O}_\varphi : C^\infty(M,\mathbb{R}) \to \mathbb{R} \ , \quad \phi \mapsto \mathcal{O}_\varphi(\phi) = \int_M \mathrm{vol}\ \varphi\,\phi$$

- For the symplectic vector space $(\mathcal{E},\tau)$ we can construct a unital $(C)^*$-algebra (the CCR-algebra). The association of these algebras to objects in Man is functorial, $\mathfrak{A} : \mathrm{Man} \to {}^*\mathrm{Alg}$, and satisfies all axioms of LCQFT.

The construction of the Klein-Gordon functor in pictures

Category Man ⇜ Interpretation: Physical backgrounds

↓ Phasespace functor $\mathfrak{PhSp}$

Category Symp ⇜ Interpretation: Labels for functionals $\mathcal{O}_\varphi$

↓ CCR-functor $\mathfrak{CCR}$

Category *Alg ⇜ Interpretation: Quantized functionals $\widehat{\mathcal{O}}_\varphi$

- This two step procedure, slightly modified for allowing also configurations in vector bundles, can be applied sucessfully to quantize all linear bosonic (and also fermionic) hyperbolic theories [Bär, Ginoux, Pfäffle].
- Also all affine theories, i.e. the configuration bundle is an affine bundle, can be quantized by extending these techniques [Benini, Dappiaggi, AS : 1210.3457].
- However, in gauge theories the construction of the first functor $\mathfrak{PhSp}$ is complicated due to the presence of gauge invariance. In the next part of my talk I will show how to deal with these complications.

Quantized Abelian principal connections

Some (potential) problems and their solution

- **Problem 1:** Connection forms ω live on principal bundles, not on spacetimes.
  - $\rightarrow$ Replace the category Man by the category PrBu:

    $\mathrm{Obj}(\mathsf{PrBu})$ are triples (M, G, P), where M is an object in Man, G is a connected Abelian Lie group with bi-invariant pseudo-Riemannian metric and P is a principal G-bundle over M.

    $\mathrm{Mor}(\mathsf{PrBu})$ are principal bundle maps $F = (f : P_1 \to P_2, \phi : G_1 \to G_2)$, such that ϕ is an isometry and $\underline{f} : M_1 \to M_2$ is a morphism in Man.

- **Problem 2:** The space of connections $\mathrm{Con}(P)$ is **not** a vector space, but an affine space over $\Omega^1(M, \mathfrak{g})$.
  - $\rightarrow$ Using Atiyah's sequence, we can realize $\mathrm{Con}(P)$ as the space of sections of an affine bundle, the bundle of connections. With the tools developed in [Benini, Dappiaggi, AS : 1210.3457] we can get a hand on QFTs on affine bundles.

- **Problem 3:** The equation of motion $\delta \mathcal{F}_\omega = 0$ is not hyperbolic.
  - $\rightarrow$ With the interpretation of φ as labels of suitable functionals $\mathcal{O}_\varphi$ we can restrict to the space of gauge invariant elements $\mathcal{E}^{\mathrm{inv}}$. We shall show that on $\mathcal{E}^{\mathrm{inv}}$ (modulo equation of motion) there is a canonical **pre**symplectic structure τ.

- **Problem 4:** Why is there a null-space of τ?
  - $\rightarrow$ This will be a surprise and is connected to topological quantum fields.

The Atiyah sequence and the bundle of connections

◇ To any object in PrBu we can associate the Atiyah sequence

$$M \times \{0\} \longrightarrow \mathrm{ad}(P) \xrightarrow{\iota} TP/G \underset{\lambda}{\overset{\pi_*}{\rightleftarrows}} TM \longrightarrow M \times \{0\}$$

◇ It is not too hard to show that splittings $\lambda : TM \to TP/G$, i.e. $\pi_\star \circ \lambda = \mathrm{id}_{TM}$, are in bijective correspondence with connection forms on P.

◇ These splittings can be described as sections of an affine subbundle $\mathcal{C}(P) \subseteq \mathrm{Hom}(TM, TP/G)$, the bundle of connections. The underlying vector bundle is $\mathrm{Hom}(TM, \mathrm{ad}(P))$.

NB: The set of sections $\Gamma^\infty(M, \mathcal{C}(P))$ is an affine space over the vector space $\Gamma^\infty(M, \mathrm{Hom}(TM, \mathrm{ad}(P))) \simeq \Omega^1(M, \mathfrak{g})$. We denote for simplicity the action by $\lambda + \eta$.

Lem: $\mathrm{Con}(P)$ and $\Gamma^\infty(M, \mathcal{C}(P))$ are isomorphic as affine spaces over $\Omega^1(M, \mathfrak{g})$.

☺ We have a nice configuration bundle (over M) for principal connections.

The vector dual bundle and observables

- Since $\mathcal{C}(P) \stackrel{\pi_{\mathcal{C}(P)}}{\longrightarrow} M$ is an affine bundle, it has a **vector** dual bundle $\mathcal{C}(P)^\dagger \stackrel{\pi_{\mathcal{C}(P)^\dagger}}{\longrightarrow} M$. (The fibres $\mathcal{C}(P)^\dagger|_x$ are the affine maps from $\mathcal{C}(P)|_x$ to $\mathbb{R}$.)
- To any $\varphi \in \Gamma^\infty_0(M, \mathcal{C}(P)^\dagger)$ we associate a functional

$$\mathcal{O}_\varphi : \Gamma^\infty(M, \mathcal{C}(P)) \to \mathbb{R} \ , \quad \lambda \mapsto \mathcal{O}_\varphi(\lambda) = \int_M \mathrm{vol}\ \varphi(\lambda)$$

- These functionals are (of course) **not** linear, but they are affine

$$\mathcal{O}_\varphi(\lambda + \eta) = \mathcal{O}_\varphi(\lambda) + \underbrace{\int_M \varphi_V \wedge *(\eta)}_{=:\langle \varphi_V, \eta \rangle} \ , \quad \underbrace{\varphi_V \in \Omega^1_0(M, \mathfrak{g}^*)}_{\text{linear part of } \varphi}$$

NB: There exist trivial functionals $\mathcal{O}_\varphi(\lambda) = 0$, for all λ, iff

$$\varphi \in \mathrm{Triv} := \Big\{ \varphi \in \Gamma^\infty_0(M, \mathcal{C}(P)^\dagger) : \exists a \in C^\infty_0(M), \int_M \mathrm{vol}\ a = 0, \varphi = a\, \mathbb{1} \Big\}$$

$\Rightarrow$ A good set of labels for functionals is $\Gamma^\infty_0(M, \mathcal{C}(P)^\dagger)/\mathrm{Triv}$.

Gauge invariance

◇ The gauge group $\mathrm{Gau}(P)$ is isomorphic to $C^\infty(M,G)$, and $\widehat{f} \in C^\infty(M,G)$ acts on $\lambda \in \Gamma^\infty(M,\mathcal{C}(P))$ by $f^*(\lambda) := \lambda + \widehat{f}^*(\mu_G)$.

Def: (i) We define the Abelian subgroup $\mathrm{d}C^\infty(M,\mathfrak{g}) \subseteq B_G := \{\widehat{f}^*(\mu_G) : \widehat{f} \in C^\infty(M,G)\} \subseteq \Omega^1_{\mathrm{d}}(M,\mathfrak{g})$.

(ii) We say that $\varphi \in \Gamma^\infty_0(M,\mathcal{C}(P)^\dagger)$ is gauge invariant, if for all λ, $\mathcal{O}_\varphi(\lambda + B_G) = \mathcal{O}_\varphi(\lambda) + \langle \varphi_V, B_G\rangle = \mathcal{O}_\varphi(\lambda)$. The set of gauge invariant elements is denoted by $\mathcal{E}^{\mathrm{inv}} \subseteq \Gamma^\infty_0(M,\mathcal{C}(P)^\dagger)/\mathrm{Triv}$.

Prop: a) $B_G/\mathrm{d}C^\infty(M,\mathfrak{g}) \subseteq H^1_{\mathrm{dR}}(M,\mathfrak{g})$ is isomorphic to $\check{H}^1(M,2\pi i\,\mathbb{Z})^{\oplus k}$, where k comes from the isomorphism $G \simeq \mathbb{T}^k \times \mathbb{R}^l$.

b) $\mathcal{E}^{\mathrm{inv}} = \big\{\varphi \in \Gamma^\infty_0(M,\mathcal{C}(P)^\dagger)/\mathrm{Triv} : \varphi_V \in \delta\Omega^2_0(M,i\,\mathbb{R})^{\oplus k} \oplus \Omega^1_{0,\,\delta}(M,\mathbb{R})^{\oplus l}\big\}$

NB: There happened something strange:

- The actual gauge invariance condition is $\langle \varphi_V, \check{H}^1(M,2\pi i\,\mathbb{Z})^{\oplus k}\rangle = \{0\}$.
- For M of finite-type (this is what we assume) the universal coefficient theorem and the de Rham iso tells us that $\check{H}^1(M,2\pi i\,\mathbb{Z})^{\oplus k} \otimes_{\mathbb{Z}} \mathbb{R} \simeq H^1_{\mathrm{dR}}(M,i\mathbb{R})^{\oplus k}$.
- Linearity implies $\langle \varphi_V, H^1_{\mathrm{dR}}(M,i\mathbb{R})^{\oplus k}\rangle = \{0\}$.

☹ $\{\mathcal{O}_\varphi : \varphi \in \mathcal{E}^{\mathrm{inv}}\}$ is **not** separating on gauge equivalence classes of connections!

☺ I will show how to repair this at the end of my talk...

The dynamics

- Let us try to understand Maxwell's equation $\delta \mathcal{F}_\omega = 0$ in detail.
- The curvature can be regarded as an affine differential operator

$$\mathcal{F} : \Gamma^\infty(M, \mathcal{C}(P)) \to \Omega^2(M, \mathfrak{g}) \ , \quad \lambda \mapsto \mathcal{F}(\lambda) := \mathcal{F}_{\omega_\lambda}$$

with linear part $\mathcal{F}_V : \Omega^1(M, \mathfrak{g}) \to \Omega^2(M, \mathfrak{g})\,,\ \eta \mapsto -\mathrm{d}\eta$, i.e. $\mathcal{F}(\lambda + \eta) = \mathcal{F}(\lambda) - \mathrm{d}\eta$.

- Also the Maxwell operator $\mathsf{MW} := \delta \circ \mathcal{F} : \Gamma^\infty(M, \mathcal{C}(P)) \to \Omega^1(M, \mathfrak{g})$ is an affine differential operator, $\mathsf{MW}(\lambda + \eta) = \mathsf{MW}(\lambda) - \delta\mathrm{d}\eta$.

Prop: Any affine differential operator $P : \Gamma^\infty(M, \mathsf{A}) \to \Gamma^\infty(M, \mathsf{W})$ is formally adjoinable to a differential operator $P^* : \Gamma^\infty_0(M, \mathsf{W}^*) \to \Gamma^\infty_0(M, \mathsf{A}^\dagger)$.
When regarded as a linear map $P^* : \Gamma^\infty_0(M, \mathsf{W}^*) \to \Gamma^\infty_0(M, \mathsf{A}^\dagger)/\mathrm{Triv}$, then P^* is unique.

Cor: (i) $\mathcal{F}^* : \Omega^2_0(M, \mathfrak{g}^*) \to \Gamma^\infty_0(M, \mathcal{C}(P)^\dagger)/\mathrm{Triv}$
(ii) $\mathsf{MW}^* = \mathcal{F}^* \circ \mathrm{d} : \Omega^1_0(M, \mathfrak{g}^*) \to \Gamma^\infty_0(M, \mathcal{C}(P)^\dagger)/\mathrm{Triv}$

$\Rightarrow$ On-shell functionals are labeled by the quotient $\mathcal{E} := \mathcal{E}^{\mathrm{inv}}/\mathsf{MW}^*[\Omega^1_0(M, \mathfrak{g}^*)]$

The presymplectic structure

◇ To obtain a phasespace, we have to equip $\mathcal{E} := \mathcal{E}^{\mathrm{inv}}/\mathsf{MW}^*[\Omega^1_0(M,\mathfrak{g}^*)]$ with a (pre-)symplectic structure $\tau : \mathcal{E}\times\mathcal{E}\to\mathbb{R}$.

Prop: Let $G_{(1)} : \Omega^1_0(M,\mathfrak{g}) \to \Omega^1(M,\mathfrak{g})$ be the causal propagator for the d'Alembert operator $\Box_{(1)} = \delta\circ\mathrm{d} + \mathrm{d}\circ\delta$ on $\Omega^1(M,\mathfrak{g})$. Then the bilinear map

$$\tau\big([\varphi],[\psi]\big) := \langle \varphi_V, G_{(1)}(\psi_V)\rangle_h := \int_M \varphi_V \wedge *\big(G_{(1)}(h^{-1}(\psi_V))\big)$$

is a presymplectic structure on $\mathcal{E}$.

NB: Note that τ only acts on the linear parts φ_V,ψ_V! τ is exactly the "bracket" that follows from a generalization to gauge theories of Peierls' derivation applied to the Lagrangian $\mathcal{L}[\lambda] = -\frac{1}{2}h\big(\mathcal{F}(\lambda)\big)\wedge *\big(\mathcal{F}(\lambda)\big)$.

Thm: Denote by $\mathcal{N}\subseteq\mathcal{E}$ the null-space, $\tau([\psi],\mathcal{E}) = \{0\}$ iff $[\psi]\in\mathcal{N}$. Then

$$\mathcal{N} = \{\psi\in\mathcal{E}^{\mathrm{inv}} : h^{-1}(\psi_V)\in\delta\Omega^2_{0,\mathrm{d}}(M,i\mathbb{R})^{\oplus k}\oplus\delta\big(\Omega^2_0(M,\mathbb{R})\cap\mathrm{d}\Omega^1_{\mathrm{tc}}(M,\mathbb{R})\big)^{\oplus l}\}/\mathsf{MW}^*[\Omega^1_0(M,\mathfrak{g}^*)]$$

Cor: For $G = U(1)$, $\mathcal{N} = \big\{[\varphi]\in\mathcal{E} : [\varphi_V]\in\delta H^2_{0\,\mathrm{dR}}(M,\mathfrak{g}^*)\big\}$.

⇒ Topology dependent null-space!?! Who ordered that?

The functor

- Let us define the categories $\mathsf{PreSymp}^{(\mathrm{ni})}$:

  $\mathrm{Obj}(\mathsf{PreSymp}^{(\mathrm{ni})})$ are presymplectic vector spaces

  $\mathrm{Mor}(\mathsf{PreSymp}^{(\mathrm{ni})})$ are (not necessarily) injective linear maps preserving the presymplectic structures

Thm:
(i) Associating the presymplectic vector space $(\mathcal{E}, \tau)$ constructed before (and appropriate push-forwards of $\mathrm{Mor}(\mathsf{PrBu})$), we obtain a covariant functor $\mathfrak{PhSp} : \mathsf{PrBu} \to \mathsf{PreSymp}^{\mathrm{ni}}$.

(ii) Composing with the CCR-functor $\mathfrak{CCR} : \mathsf{PreSymp}^{(\mathrm{ni})} \to {}^*\mathsf{Alg}^{(\mathrm{ni})}$ leads to a covariant functor $\mathfrak{A} := \mathfrak{CCR} \circ \mathfrak{PhSp} : \mathsf{PrBu} \to {}^*\mathsf{Alg}^{\mathrm{ni}}$ that satisfies the causality property and the time-slice axiom.

(iii) Fixing $G = U(1)$ in the category PrBu, we can restrict to a covariant functor $\mathfrak{A} : \mathsf{PrBu}^{U(1)} \to {}^*\mathsf{Alg}^{\mathrm{ni}}$ that describes the quantized Maxwell field.

NB: Neither the functor $\mathfrak{A} : \mathsf{PrBu} \to {}^*\mathsf{Alg}^{\mathrm{ni}}$, nor its restriction to $\mathsf{PrBu}^{U(1)}$ or $\mathsf{PrBu}^{\mathbb{R}}$, are functors to ${}^*\mathsf{Alg}$. Injectivity is violated e.g. for the embedding

$$\mathbb{R}^{1,n} \setminus J(\{0\}) \hookrightarrow \mathbb{R}^{1,n}$$

of a globally hyperbolic submanifold into the Minkowski spacetime.

Topological quantum fields in AQFT

Topological quantum fields: $G = U(1)$

◇ According to [Brunetti, Fredenhagen, Verch] a locally covariant quantum field is a natural transformation $\Psi = \{\Psi_M\} : \mathfrak{D} \Rightarrow \mathfrak{A}$ from a covariant functor $\mathfrak{D}$: Man $\to$ Vec describing test sections to the QFT functor $\mathfrak{A}$: Man $\to$ *Alg.

Def: (i) A generally covariant topological quantum field is a natural transformation $\Psi : \mathfrak{T} \Rightarrow \mathfrak{A}$ from a covariant functor $\mathfrak{T} : \mathsf{PrBu}^{U(1)} \to \mathsf{Vec}$ describing "topological information" to the QFT functor $\mathfrak{A} : \mathsf{PrBu}^{U(1)} \to {}^*\mathsf{Alg}^{\mathrm{ni}}$.

(ii) Denote by $\mathfrak{H}_2, \mathfrak{H}_{-2} : \mathsf{PrBu}^{U(1)} \to \mathsf{Vec}$ the singular homology functors with $\mathfrak{H}_2(M,G,P) = H_2(M,\mathfrak{g}^*)$ and $\mathfrak{H}_{-2}(M,G,P) = H_{\dim(M)-2}(M,\mathfrak{g}^*)$.

(iii) Let $\mathcal{K} : H_p(M,\mathfrak{g}^*) \to H^p_{0\,\mathrm{dR}}(M,\mathfrak{g}^*)$ be the isomorphism induced by Poincaré duality and the de Rham isomorphism.

Thm: The collections $\{\Psi^{\mathrm{mag}}_{(M,G,P)}\}$ and $\{\Psi^{\mathrm{el}}_{(M,G,P)}\}$ of linear maps

a) $\Psi^{\mathrm{mag}}_{(M,G,P)} : \mathfrak{H}_2(M,G,P) \to \mathfrak{A}(M,G,P)\ , \quad \sigma \mapsto \big[\mathcal{F}^*\big(\mathcal{K}(\sigma)\big)\big]$

b) $\Psi^{\mathrm{el}}_{(M,G,P)} : \mathfrak{H}_{-2}(M,G,P) \to \mathfrak{A}(M,G,P)\ , \quad \sigma \mapsto \big[\mathcal{F}^*\big(*\big(\mathcal{K}(\sigma)\big)\big)\big]$

are generally covariant topological quantum fields.

NB: Both Ψ^{mag} and Ψ^{el} map to the center of the algebras! Looking at the corresponding classical observables $\mathcal{O}_\varphi$ we see that Ψ^{mag} measures the magnetic charge (Euler class of P) $[\mathcal{F}(\lambda)] \in H^2_{\mathrm{dR}}(M,\mathfrak{g})$ and Ψ^{el} the electric charge $[*\mathcal{F}(\lambda)] \in H^{\dim(M)-2}_{\mathrm{dR}}(M,\mathfrak{g})$.

Charge-zero functor is a locally covariant QFT

The charge-zero functor and the locality property

- **Physics insight:** If I live in a universe only containing electromagnetism and gravity, then all electric charge measurements yield zero.
- **Mathematical realization:** Instead of taking the quotient of $\mathcal{E}^{\mathrm{inv}}$ by $\mathsf{MW}^*[\Omega^1_0(M,\mathfrak{g}^*)]$, we should take the quotient by $\mathcal{F}^*[\Omega^2_{0,\mathrm{d}}(M,\mathfrak{g}^*)]$ which contains the electric charges **and** the EOM!

Prop: On $\mathcal{E}^0 := \mathcal{E}^{\mathrm{inv}}/\mathcal{F}^*[\Omega^2_{0,\mathrm{d}}(M,\mathfrak{g}^*)]$ there is a presymplectic structure

$$\tau^0\big([\varphi],[\psi]\big) := \langle \varphi_V, G_{(1)}(\psi_V)\rangle_h = \int_M \varphi_V \wedge *\big(G_{(1)}\big(h^{-1}(\psi_V)\big)\big) \ .$$

The corresponding null-space is $\mathcal{N}^0 = \big[\{\psi \in \mathcal{E}^{\mathrm{inv}} : \psi_V = 0\}\big]$.

Thm: (i) Associating the presymplectic vector space $(\mathcal{E}^0,\tau^0)$ (and appropriate push-forwards of $\mathrm{Mor}(\mathsf{PrBu})$), we obtain a covariant functor $\mathfrak{PhSp}^0 : \mathsf{PrBu}^{U(1)} \to \mathsf{PreSymp}$.

(ii) Composing with the CCR-functor $\mathfrak{CCR} : \mathsf{PreSymp} \to {}^*\mathsf{Alg}$ leads to a covariant functor $\mathfrak{A}^0 := \mathfrak{CCR}\circ\mathfrak{PhSp}^0 : \mathsf{PrBu}^{U(1)} \to {}^*\mathsf{Alg}$ that satisfies the causality property, the time-slice axiom **and** the injectivity (locality) property.

NB: There exists a nontrivial generally covariant topological quantum field $\Psi^{\mathrm{mag}} : \mathfrak{H}_2 \Rightarrow \mathfrak{A}^0$ measuring the magnetic charge (Euler class of P).

Towards a very good functor for the Maxwell field

Remaining problems with our functors

- Our first functor $\mathfrak{A} : \mathsf{PrBu}^{U(1)} \to {}^*\mathsf{Alg}^{\mathrm{ni}}$, while being quite good, has two problems:
  - ☹ Injectivity property ($\widehat{=}$ locality) of LCQFT is violated.
  - ☹ Classical observables $\{\mathcal{O}_\varphi : \varphi \in \mathcal{E}\}$ are not separating on gauge equivalence classes of connections.
- Our second functor $\mathfrak{A}^0 : \mathsf{PrBu}^{U(1)} \to {}^*\mathsf{Alg}$, while being good, still has one problem:
  - ☺ All axioms of LCQFT hold.
  - ☹ Classical observables $\{\mathcal{O}_\varphi : \varphi \in \mathcal{E}\}$ are not separating on gauge equivalence classes of connections.
- So we **should try** to construct third functor $\mathfrak{A}' : \mathsf{PrBu}^{U(1)} \to {}^*\mathsf{Alg}$ which is very good:
  - ☺ All axioms of LCQFT hold.
  - ☺ Classical observables $\{\mathcal{W}_\varphi : \varphi \in \mathcal{E}'\}$ are separating on gauge equivalence classes of connections.

Options for the phasespaces for the very good functor

1. **Wilson loops:** Let $\gamma : S^1 \to M$ be a loop.

$$T_\gamma : \Gamma^\infty(M, \mathcal{C}(P)) \to \mathbb{C} \;, \quad \lambda \mapsto \mathrm{Tr}\big(h_\gamma(\lambda)\big)$$

☺ Wilson loop functionals are separating on gauge equivalence classes of connections.

☹ Too singular: $\{T_\gamma, T_{\gamma'}\}_{\mathrm{Peierls}} = \infty$ if γ, γ' connected by causal curve.

2. **Wilson tubes:** [Baez] Let $\Gamma : S^1 \times D^{n-1} \to M$ and $\omega \in \Omega_0^{n-1}(D^{n-1})$.

$$T_\Gamma : \Gamma^\infty(M, \mathcal{C}(P)) \to \mathbb{C} \;, \quad \lambda \mapsto \int_{D^{n-1}} \omega(x) \mathrm{Tr}\big(h_{\Gamma(\cdot,x)}(\lambda)\big)$$

☺ Wilson tube functionals are separating on gauge equivalence classes of connections.

☹ Too singular: $\{T_\Gamma, T_{\Gamma'}\}_{\mathrm{Peierls}} = \text{finite}$, but $\star$-product is ∞.

3. **Wilson Wursts:** Take exponentials of $\mathcal{O}_\varphi$ observables

$$\mathcal{W}_\varphi : \Gamma^\infty(M, \mathcal{C}(P)) \to \mathbb{C} \;, \quad \lambda \mapsto e^{2\pi i \, \mathcal{O}_\varphi(\lambda)}$$

☺ Obviously regular! Checking the details we also obtain separating condition!

Properties of Wilson Wursts

◇ Due to taking the exponential of $\mathcal{O}_\varphi$, $\varphi \in \Gamma_0^\infty(M, \mathcal{C}(P)^\dagger)/\mathrm{Triv}$, the gauge invariance condition is weaker than before

$$\langle \varphi_V, B_G \rangle \subseteq \mathbb{Z}$$

Prop: An observable $\mathcal{W}_\varphi = e^{2\pi i\, \mathcal{O}_\varphi}$ is gauge invariant iff $\varphi \in \mathcal{E}'^{\,\mathrm{inv}}$, where

$$\mathcal{E}'^{\,\mathrm{inv}} = \Big\{ \varphi \in \Gamma_0^\infty(M, \mathcal{C}(P)^\dagger)/\mathrm{Triv} : [\varphi_V] \in \underbrace{\mathcal{D}^* \big[\mathrm{Hom}_{\mathbb{Z}}(\check{H}^1(\mathcal{U}, 2\pi i\, \mathbb{Z}), \mathbb{Z}) \big]}_{\mathcal{D}^* \text{ is dual of the Čech-de Rham iso}} \Big\} \, .$$

NB: Note that $\mathcal{E}'^{\,\mathrm{inv}}$ is **not** a vector space, but only an Abelian group!

Thm: The set of gauge invariant observables $\{\mathcal{W}_\varphi : \varphi \in \mathcal{E}'^{\,\mathrm{inv}}\}$ is separating on gauge equivalence classes of connections.

Prop: $\mathcal{E}' := \mathcal{E}'^{\,\mathrm{inv}} / \mathcal{F}^*[\Omega^2_{0,\mathrm{d}}(M, \mathfrak{g}^*)]$ is a presymplectic Abelian group with

$$\tau^0\big([\varphi], [\psi]\big) = \langle \varphi_V, G_{(1)}(\psi_V) \rangle_h = \int_M \varphi_V \wedge * \big(G_{(1)}\big(h^{-1}(\psi_V)\big)\big) \, .$$

? How can we quantize such presymplectic Abelian groups in terms of C^*-algebras? Requires generalization of [Bär, Ginoux, Pfäffle] along the lines of [Manuceau et al.]!

Conclusions and outlook

Conclusions and outlook

We have

- adapted the nice geometrical description of (Abelian) principal connections in terms of the bundle of connections to the needs of AQFT.
- constructed a covariant functor $\mathfrak{A} : \mathsf{PrBu} \to {}^*\mathsf{Alg}^{\mathrm{ni}}$ which describes quantized Abelian principal connections.
- found interesting features that we called generally covariant topological quantum fields.
- shown that setting all electric charges to zero yields a functor $\mathfrak{A}^0 : \mathsf{PrBu}^{U(1)} \to {}^*\mathsf{Alg}$ satisfying the axioms of LCQFT.
- made first progress to solve the remaining problem of our functor $\mathfrak{A}^0$.

It is still open to

- finalize the construction of the very good functor. This requires an understanding of CCR-representations of presymplectic Abelian groups like in [Manuceau et al.], which goes beyond [Bär, Ginoux, Pfäffle].
- understand if generally covariant topological quantum fields provide a way to distinguish gauge theories in AQFT.