MATH4017 Quantum Field Theory

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Chapter 1

Introduction

1.1 What is quantum field theory about?

As suggested by its name, quantum field theory (in short, QFT) is about studying the *quantum theory* of *fields*. Recall that a *field* is a concept from classical physics that is modeled by a function

$$\phi : M \longrightarrow \mathcal{Q} \ , \quad x \longmapsto \phi(x) \tag{1.1}$$

from a space or spacetime M to a set of values $\mathcal{Q}$. It is important to note that a field is a local concept, in the sense that the assigned value $\phi(x) \in \mathcal{Q}$ depends on the point $x \in M$ in space or spacetime, i.e. it might vary if we vary x. Fields come in different types, depending on what one chooses for the set of values $\mathcal{Q}$. Physical examples that you may have come across in your studies are the mass density $\rho(x)$ from continuum mechanics (this is a scalar field, i.e. $\mathcal{Q} = \mathbb{R}$), the electric $\mathbf{E}(x)$ and magnetic $\mathbf{B}(x)$ fields from electromagnetism (these are 3-dimensional vector fields, i.e. $\mathcal{Q} = \mathbb{R}^3$), and the metric field $g_{\mu\nu}(x)$ from general relativity (this is a rank 2 tensor field, i.e. $\mathcal{Q} = \mathrm{Mat}_{d\times d}(\mathbb{R})$ with d the dimension of spacetime).

Informally speaking, a *quantum field* is what one gets when applying the principles of quantum mechanics to fields. The most crucial (in my personal opinion) feature of quantum mechanics is that the values of physical quantities, such as the position and momentum of a particle, are promoted to *operators*. Different operators do in general not commute with each other, for instance the position $\hat{q}$ and momentum $\hat{p}$ operators satisfy the famous commutation relations

$$[\hat{q}, \hat{q}] = 0 = [\hat{p}, \hat{p}] \quad , \qquad [\hat{q}, \hat{p}] = \mathrm{i}\,\hbar = -[\hat{p}, \hat{q}] \quad , \tag{1.2}$$

where $[\hat{A}, \hat{B}] := \hat{A}\,\hat{B} - \hat{B}\,\hat{A}$ denotes the commutator of two operators $\hat{A}, \hat{B}$. In the Heisenberg picture, operators evolve in time $t \in \mathbb{R}$. The time evolution of an operator $\hat{A}$ is described by Heisenberg's equation

$$\frac{\mathrm{d}\hat{A}(t)}{\mathrm{d}t} = \frac{\mathrm{i}}{\hbar}\,[\hat{H}, \hat{A}(t)] \quad , \tag{1.3}$$

subject to the initial condition $\hat{A}(0) = \hat{A}$, where $\hat{H}$ denotes the Hamiltonian operator of the system. Solving Heisenberg's equation for the position operator $\hat{q}$ yields an operator valued function

$$\hat{q} \ : \ \mathbb{R} \longrightarrow \text{Operators} \ , \quad t \longmapsto \hat{q}(t) = e^{\frac{\mathrm{i}t}{\hbar}\hat{H}}\,\hat{q}\,e^{-\frac{\mathrm{i}t}{\hbar}\hat{H}} \quad , \tag{1.4}$$

which one can think of as a quantum version of the classical trajectory $q : \mathbb{R} \to \mathbb{R}\,,\ t \mapsto q(t)$ of a particle. Note that, for the typical examples of Hamiltonians such as $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$, the momentum operator $\hat{p}(t)$ can be derived by taking a t-derivative of $\hat{q}(t)$. Explicitly, one computes

$$\begin{aligned} m\,\frac{\mathrm{d}\hat{q}(t)}{\mathrm{d}t} &= \frac{\mathrm{i}\,m}{\hbar}\,[\hat{H},\hat{q}(t)] = \frac{\mathrm{i}\,m}{\hbar}\,e^{\frac{\mathrm{i}t}{\hbar}\hat{H}}\,[\hat{H},\hat{q}]\,e^{-\frac{\mathrm{i}t}{\hbar}\hat{H}} \\ &= \frac{\mathrm{i}}{2\hbar}\,e^{\frac{\mathrm{i}t}{\hbar}\hat{H}}\,[\hat{p}^2,\hat{q}]\,e^{-\frac{\mathrm{i}t}{\hbar}\hat{H}} \\ &= e^{\frac{\mathrm{i}t}{\hbar}\hat{H}}\,\hat{p}\,e^{-\frac{\mathrm{i}t}{\hbar}\hat{H}} = \hat{p}(t) \quad , \end{aligned} \tag{1.5}$$

where steps three and four follow from the commutation relations (1.2). Hence, all information about this quantum system is encoded in $\hat{q}(t)$. Since the field theoretic analog of the trajectory $q(t)$ of a particle is the field $\phi(x)$, it is natural to expect that operator valued functions

$$\hat{\phi} : M \longrightarrow \text{Operators}\ , \quad x \longmapsto \hat{\phi}(x) \tag{1.6}$$

on a space or spacetime M will play a fundamental role in QFT. We will see later in this module that this is indeed the case.

What remains unclear from what was written so far is the scope of applicability of QFT. Since the concept of a quantum field is based on that of a classical field, it is not hard to believe that one may apply these techniques to, say, the electric and magnetic fields from electromagnetism in order to obtain their quantum analogs $\hat{\mathbf{E}}(x)$ and $\hat{\mathbf{B}}(x)$. (This is indeed the case, as we shall see later in this module.) But what about particles, such as electrons? These are typically described by their trajectories $q(t)$ and not by a field, so there is no obvious classical field theory one could start from to derive a QFT for, say, electrons. This seeming incompatibility has caused a lot of confusion in the early days of QFT until it was realized that *each quantum field has an associated theory of particles*. These particles arise by studying representations of the operator valued functions $\hat{\phi}(x)$ on a Hilbert space $\mathcal{H}$ and can be interpreted as excitations of the lowest energy (i.e. vacuum) state $|0\rangle \in \mathcal{H}$ of the QFT. But that is not all: Quantum fields naturally lead to a multiparticle theory in which the particle number is not necessarily conserved. This allows for physical processes in which particles get annihilated and/or new particles get created. Recall that many important processes in nature are of this type. For example, a single neutron can transform into a triple of particles, given by a proton, an electron and an antineutrino, under a process called beta decay. This involves annihilating the neutron and creating the three other particles. Such processes do not admit a description in terms of ordinary quantum mechanics, which from the start assumes that there is a fixed number of particles that is not allowed to change over time, but they make perfectly sense in the framework of QFT. The intrinsic properties of the particles associated with a QFT, such as their masses and spins, are encoded by the type of field one considers, i.e. its underlying set of values $\mathcal{Q}$, and by its dynamics. The latter does also dictate the interactions between the particles. For example, we will see that the simplest case of a scalar field $\hat{\Phi}(x)$ will give rise to spin 0 particles and that for electrons one needs a more sophisticated field $\hat{\Psi}(x)$ whose classical set of values is the set of Dirac spinors. Furthermore, the particle associated with the electromagnetic fields $\hat{\mathbf{E}}(x)$ and $\hat{\mathbf{B}}(x)$ is the photon. These are only some of the many particles comprising the *standard model of particle physics*. It turns out that each fundamental particle (electron, quarks, neutrinos, ...) and also each fundamental force (electromagnetic, weak and strong) arises from a quantum field of a suitable type and that their interplay/interaction is governed by the dynamics resulting from an

action functional that is defined on the whole collection of fields. Unfortunately, such quantum field theoretical methods do not seem to be suitable for the gravitational field, and it is widely believed among physicists that a consistent theory of quantum gravity will go beyond the scope of QFT.

Historically, QFT was developed with the aim of providing the theoretical foundations for high-energy elementary particle physics. This endeavor has been very successful because the standard model of particle physics, which is build on the principles of QFT, describes the behavior of the elementary particles and their interactions with a remarkable accuracy. Over the past decades, the standard model got tested by more and more precise and higher energetic experiments, such as those performed at the Large Hadron Collider (LHC) at CERN, which so far have always agreed impressively well with the theoretical predictions. Because of this success, QFT has become an indispensable tool in the repertoire of every theoretical high-energy physicist. It is worthwhile to mention that the QFTs appearing in high-energy physics are defined on the Minkowski spacetime and that they are invariant under Poincaré transformations. This is because high-energy physics involves extremely fast particles, moving at almost the speed of light, which is the natural habitat of special relativity. In addition to high-energy physics, QFT has also contributed substantially to the development of other physical disciplines, such as condensed matter physics, cosmology and statistical physics. In condensed matter physics, quantum fields are used to model quasi-particles, such as phonons, magnons and others, that arise as excitations in a condensed matter system. These QFTs are usually defined on a nonrelativistic Newtonian spacetime because the velocities in a condensed matter system are small in comparison to the speed of light, however there exist materials (e.g. graphene) that require relativistic QFTs. In cosmology, the current understanding is that quantum fields in the early universe have generated quantum fluctuations that over time have grown to the galaxy structures we observe today. These QFTs are defined on curved spacetimes, in the sense of general relativity, which is an interesting topic that we however shall not touch in this module. The QFTs that arise in statistical physics are defined on Euclidean spaces, in contrast to spacetimes, and they are used to describe the probabilistic behavior of random systems.

I would like to conclude this section with some remarks about the complicated but fruitful relationship between QFT and mathematics, as this topic lies at the heart of my personal interests. Let me start with a warning: QFT is *not* (yet) a theory according to the high standards of pure mathematics. By this I mean that there does not yet exist a mathematical framework which provides an axiomatic definition of what a QFT should be and, importantly, accommodates the key examples from physics, such as the standard model of particle physics. This is also known as the *Yang-Mills Millennium Prize Problem* and whoever can solve it will be awarded $\$1,000,000$ by the Clay Mathematics Institute. Mathematicians and mathematical physicists have tried to develop such axiomatic frameworks, which started with the Wightman axioms and culminated in modern approaches such as algebraic QFT, factorization algebras and functorial QFT. This has initiated an impressive cross-fertilization between physics and many branches of mathematics, most notably operator algebras, geometry, topology and category theory. While oversimplified toy-models of QFTs that share some (but not many) similarities with physically relevant models could be constructed (in a mathematical sense) in these frameworks, it still remains an open problem to construct Yang-Mills theory and the standard model of particle physics. The way how most of the theoretical/mathematical physics literature, and also our module, deals with these issues is as follows. First of all, instead of treating QFTs in a fully general fashion, we consider only examples of QFTs in which the interactions are described *perturbatively*, i.e. we expand nonlinearities order by order in the coupling constants and typically stop after the first few terms. While there is no

mathematical reason why that should give something sensible, this approach is justified a posteriori by the fact that it matches the relevant physics with very high accuracy. (Unfortunately, nobody knows why ...) Secondly, when analyzing perturbative QFTs, one finds infinities popping up all over the place. The origin of these infinities lies in the fact that QFT deals with infinitely many degrees of freedom (at least one for each point $x \in M$, namely $\hat{\phi}(x)$) which sometimes add up to produce a divergent result in calculations. These infinities can be dealt with by redefining the parameters of the theory, which is called *renormalization*, and it turns out that this is not a bug, but rather a feature of QFT. At the end of this module you will understand what this means.

1.2 The module MATH4017

The aim of this module is to provide an introduction to QFT. We shall focus mainly on QFTs that are defined on the Minkowski spacetime, as required for applications to high-energy physics, but similar techniques apply to QFTs that are defined on other spaces or spacetimes M as well, although there are sometimes new effects such as e.g. particle creation in curved spacetimes. Since QFT is one of the cornerstones of modern physics, every course on theoretical and/or mathematical physics should expose students to this topic, and this is precisely what we are going to do. Due to the immense relevance of QFT in various areas of physics, as well as its rich and fruitful interplay with modern mathematics, see Section 1.1, it is likely that you will benefit from this module in your future career, whether this will be in physics, mathematics or something else.

The way how we approach QFT is by focusing first on the simplest examples in order to understand the underlying mechanisms. After a brief introduction to classical field theory, we study the quantization of the free Klein-Gordon field on the Minkowski spacetime, which is a scalar field that satisfies a linear equation of motion. In this context we shall substantiate the claim from Section 1.1 that quantum fields are operator valued functions $\hat{\Phi}(x)$ whose representations on a Hilbert space $\mathcal{H}$ give rise to a concept of particles. For the free Klein-Gordon quantum field, these particles do not interact with each other as a consequence of linearity of the equation of motion. We shall then study more sophisticated examples of quantum scalar fields that display nonlinear features and thereby lead to interactions between particles. Treating these nonlinearities perturbatively, we will derive graphical techniques, called Feynman diagrams and Feynman rules, that allow us to compute transition amplitudes for particle interactions, such as scattering processes and particle decays.

After understanding the foundations of QFT for the simplest case of scalar fields, we will spend some time to generalize our techniques to other types of fields that appear in nature. Most notably, we will develop the free quantum Dirac field, which includes the electron as an example, and also study the quantization of the electromagnetic field, leading to photons. These new examples will come with new difficulties that have to be overcome, but this will pay off because there will be interesting new features.

After the basics are settled, we will dive into more advanced topics in QFT. The first goal is to make sense of the infinities we encounter at higher orders in perturbation theory, which leads us to the development of renormalization techniques. The second goal is to understand the structure of the standard model of particle physics, in particular its particle content and their interactions. Additional advanced topics will be discussed during the student projects/presentations at the end of the second semester.

1.3 Lecture notes and further literature

These lecture notes are intended to serve as a starting point for your exploration of QFT. I will try to explain the relevant concepts in a concise and pragmatic way, which necessarily leads to some omissions and shortcuts in comparison to more complete textbooks on QFT. Furthermore, these lecture notes reflect my personal taste and point of view on the subject, which very likely differs from that of, say, a high-energy physicist, a condensed matter theorist or a cosmologist. In order to get a broader context and understanding of QFT, it is therefore recommended, and in the later parts of the module even required, that you use additional literature. There are zillions of QFT books on the market, so you will probably ask which one is best. There is no definitive answer to this question, because what is best for you will heavily depend on your scientific background, interests and personal preferences. So it is recommended that you explore different books by yourself. A collection of open access books for high-energy physics is available here.

As I do not want to leave you with an empty reference list, let me list a few books with which I have interacted more recently or as a student. I will add some comments that might help you to decide if this book is suitable for you.

[1] M. Maggiore, *A Modern Introduction to Quantum Field Theory*, Oxford University Press (2005).

This is the book that I have used as a student in my first QFT module. It covers most of the basic topics of our module, but it is a bit too thin on the more advanced topics.

[2] M. E. Peskin and D. V. Schroeder, *An Introduction To Quantum Field Theory*, CRC Press (1995).

This is one of traditional and widely used QFT books. It covers the content of this module and much more. I personally like the chapters about renormalization.

[3] W. Greiner and J. Reinhardt, *Field Quantization*, Springer Berlin/Heidelberg (1996).

This is a nice book that covers the foundations of QFT very well and in much detail. Unfortunately, it does not discuss more advanced topics, such as renormalization and the standard model of particle physics.

[4] H. Nastase, *Introduction to Quantum Field Theory*, Cambridge University Press (2019).

This is a well-written book that covers the content of this module and much more. It has the advantage that it develops both the operator and the path-integral approach to QFT.

[5] M. Srednicki, *Quantum Field Theory*, Cambridge University Press (2007). Available online on the author's website and it also comes with written solutions to problems

This is a well-written book that covers most of the content of this module and much more. It is chopped into short sections about individual topics, which clearly indicate the required prerequisites at the beginning. Unfortunately, it focuses mainly on the path-integral approach and keeps the operator approach to a bare minimum.

[6] M. D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press (2013).

This is a nice book that covers many details of the standard model of particle physics.

1.4 Notations and conventions

Warning 1.1. Different QFT textbooks use different notations and conventions, especially when it comes to special relativity and Fourier transforms. Below I will list the conventions that are used in these lecture notes.

Quantum mechanics:

- We work in natural units in which the (reduced) Planck constant is one, i.e. $\hbar = 1$.
- We do *not* use hats or other decorations to denote operators. It will be clear from the context if a symbol such as A denotes an operator or a classical quantity.
- The commutator between two operators A and B is defined as
$$[A, B] := A\,B - B\,A \quad . \tag{1.7}$$
- The canonical commutation relations (CCRs) between position q_a and momentum p^a operators, where $a = 1, \ldots, N$ is an index running over the degrees of freedom, are given by
$$[q_a, q_b] = 0 = [p^a, p^b] \quad , \qquad [q_a, p^b] = \mathrm{i}\,\delta_a^b = -[p^b, q_a] \quad , \tag{1.8}$$
where δ_a^b denotes the Kronecker delta, i.e. $\delta_a^b = 1$ for $a = b$ and $\delta_a^b = 0$ else.
- States in a Hilbert space are denoted by Dirac's ket notation $|\psi\rangle \in \mathcal{H}$ and inner products by the bra-ket notation $\langle\phi|\psi\rangle$. The matrix-element of an operator A between two states $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ will be denoted by
$$\langle\phi|A|\psi\rangle := \langle\phi|A\psi\rangle \quad . \tag{1.9}$$
- The adjoint of an operator A will be denoted by $A^\dagger$. By definition, one has
$$\langle A^\dagger\phi|\psi\rangle = \langle\phi|A\psi\rangle = \langle\phi|A|\psi\rangle \quad , \tag{1.10}$$
for all states $|\psi\rangle, |\phi\rangle \in \mathcal{H}$. In particular, one can read the matrix-element $\langle\phi|A|\psi\rangle$ in two different but equivalent ways: 1.) The operator A acts from left to right on the ket-state $|\psi\rangle$, which yields $A|\psi\rangle = |A\psi\rangle$ and hence $\langle\phi|A\psi\rangle$. 2.) The operator A acts from right to left on the bra-state $\langle\phi|$, which requires taking the adjoint $\langle\phi|A = \langle A^\dagger\phi|$ and yields $\langle A^\dagger\phi|\psi\rangle$.
- We work mostly in the Heisenberg picture, where states are time-independent. The time evolution of an operator A is described by Heisenberg's equation
$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \mathrm{i}\,[H, A(t)] \quad , \tag{1.11}$$
subject to the initial condition $A(0) = A$, where H is the Hamiltonian operator of the system. The solution of this equation is
$$A(t) = e^{\mathrm{i}t[H,-]}\,A = e^{\mathrm{i}Ht}\,A\,e^{-\mathrm{i}Ht} \quad . \tag{1.12}$$

Special relativity:

- We work in natural units in which the speed of light is one, i.e. $c = 1$.
- Our conventions for the d-dimensional Minkowski spacetime $(\mathbb{R}^d, \eta)$ are

$$x = \begin{pmatrix} x^0 \\ x^1 \\ \vdots \\ x^{d-2} \\ x^{d-1} \end{pmatrix} = \begin{pmatrix} t \\ x^1 \\ \vdots \\ x^{d-2} \\ x^{d-1} \end{pmatrix} \in \mathbb{R}^d \quad , \qquad \eta = \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in \mathrm{Mat}_{d\times d}(\mathbb{R}) \quad , \qquad (1.13)$$

  i.e. we use the 'mostly plus' convention for the Minkowski metric. The dimension d is sometimes taken to be 4, i.e. one time dimension and three space dimensions as observed in nature, but we will also consider examples of QFTs in other dimensions.

- We will abbreviate the collection of space coordinates by boldface letters, i.e. we write

$$x = \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} := \begin{pmatrix} t \\ x^1 \\ \vdots \\ x^{d-2} \\ x^{d-1} \end{pmatrix} \in \mathbb{R}^d \quad . \qquad (1.14)$$

- We will often use index notation and write x^μ for the spacetime coordinates and $\eta_{\mu\nu}$ for the Minkowski metric. Greek letters $\mu, \nu, \rho, \ldots$ from the middle of the alphabet will run over the whole range $0, 1, \ldots, d-1$ and, by Einstein's summation convention, summations over repeated upper and lower indices will be suppressed. For the space coordinates $\mathbf{x}$, we use the index notation x^i, where Latin letters $i, j, k, \ldots$ from the middle of the alphabet will run over the range $1, 2, \ldots, d-1$. The summation convention will also be used for such spatial indices.
- The inverse Minkowski metric $\eta^{\mu\nu}$ is defined by

$$\eta^{\mu\nu}\,\eta_{\nu\rho} = \delta^\mu_\rho = \eta_{\rho\nu}\,\eta^{\nu\mu} \quad . \qquad (1.15)$$

  Its associated matrix η^{-1} has the same form as η.

- A Poincaré transformation is a coordinate transformation of the form

$$x'^\mu = \Lambda^\mu{}_\nu\, x^\nu + b^\mu \quad , \qquad (1.16)$$

  where b^μ is a constant d-vector and $\Lambda^\mu{}_\nu$ is a constant $d \times d$-matrix that satisfies the condition

$$\eta_{\mu\nu}\,\Lambda^\mu{}_\rho\,\Lambda^\nu{}_\sigma = \eta_{\rho\sigma} \quad . \qquad (1.17)$$

For $b^\mu = 0$, one calls $x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$ a Lorentz transformation. Unless stated otherwise, we shall always assume that $\det(\Lambda) = +1$ and that the entry $\Lambda^0{}_0 > 0$ is positive. This special class of Lorentz/Poincaré transformations is called *proper* and *orthochronous*. It excludes for example the time-reversal transformation $(t, \mathbf{x}) \mapsto (-t, \mathbf{x})$, which is studied separately in QFT.

- Defining

$$\Lambda_\mu{}^\nu := \eta_{\mu\rho}\, \eta^{\nu\sigma}\, \Lambda^\rho{}_\sigma \tag{1.18}$$

in terms of raising and lowering indices, one obtains a new matrix that satisfies

$$\eta^{\mu\nu}\, \Lambda_\mu{}^\rho\, \Lambda_\nu{}^\sigma = \eta^{\rho\sigma} \quad . \tag{1.19}$$

- Tangent vectors v^μ and cotangent vectors h_μ on the Minkowski spacetime transform under Poincaré transformations as

$$v'^\mu = \Lambda^\mu{}_\nu\, v^\nu \quad , \qquad h'_\mu = \Lambda_\mu{}^\nu\, h_\nu \quad . \tag{1.20}$$

Note that, in contrast to the coordinates x^μ, there is no translation term b^μ in these transformation formulas. The following contractions

$$h\, v := h_\mu\, v^\mu \quad , \qquad v\, w := \eta_{\mu\nu}\, v^\mu\, w^\nu \quad , \qquad h\, l := \eta^{\mu\nu}\, h_\mu\, l_\nu \tag{1.21}$$

among tangent vectors v^μ, w^μ and cotangent vectors h_μ, l_μ are Poincaré invariant. Splitting tangent and cotangent vectors into their time and space parts, these contractions read as

$$h\, v = h_0\, v^0 + \mathbf{h}\, \mathbf{v} = h_0\, v^0 + h_i\, v^i \quad , \tag{1.22a}$$

$$v\, w = -v^0\, w^0 + \mathbf{v}\, \mathbf{w} = -v^0\, w^0 + \delta_{ij}\, v^i\, w^j \quad , \tag{1.22b}$$

$$h\, l = -h_0\, l_0 + \mathbf{h}\, \mathbf{l} = -h_0\, l_0 + \delta^{ij}\, h_i\, l_j \quad , \tag{1.22c}$$

where the minus signs arise from $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$.

Fourier transform:

- We define the Fourier transform of a function f on an n-dimensional Cartesian space $\mathbb{R}^n$ with coordinates $\mathbf{x} = (x^1, \ldots, x^n)$ by

$$\widetilde{f}(\mathbf{k}) := \int_{\mathbb{R}^n} f(\mathbf{x})\, e^{-\mathrm{i}\,\mathbf{k}\,\mathbf{x}}\, \mathrm{d}\mathbf{x} \quad , \tag{1.23}$$

where $\mathbf{k} = (k_1, \ldots, k_n)$ denote the Fourier momenta, $\mathbf{k}\,\mathbf{x} := k_i\, x^i$ is obtained by summation over $i = 1, \ldots, n$, and $\mathrm{d}\mathbf{x} := \mathrm{d}x^1 \cdots \mathrm{d}x^n$ is the n-dimensional volume element.

- The Fourier transform can be inverted via the inverse Fourier transform

$$f(\mathbf{x}) = \int_{\mathbb{R}^n} \widetilde{f}(\mathbf{k})\, e^{\mathrm{i}\,\mathbf{k}\,\mathbf{x}}\, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^n} \quad . \tag{1.24}$$

In fact, inserting (1.23) into (1.24), we compute

$$\begin{aligned}\int_{\mathbb{R}^n} \widetilde{f}(\mathbf{k})\, e^{\mathrm{i}\,\mathbf{k}\,\mathbf{x}}\, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^n} &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{y})\, e^{\mathrm{i}\,\mathbf{k}\,(\mathbf{x}-\mathbf{y})}\, \mathrm{d}\mathbf{y}\, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^n} \\ &= \int_{\mathbb{R}^n} f(\mathbf{y}) \left(\int_{\mathbb{R}^n} e^{-\mathrm{i}\,\mathbf{k}\,(\mathbf{y}-\mathbf{x})}\, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^n} \right) \mathrm{d}\mathbf{y} \\ &= \int_{\mathbb{R}^n} f(\mathbf{y})\, \delta(\mathbf{y}-\mathbf{x})\, \mathrm{d}\mathbf{y} = f(\mathbf{x}) \quad , \end{aligned} \tag{1.25}$$

where in the third step we have used that

$$\delta(\mathbf{y}-\mathbf{x}) = \int_{\mathbb{R}^n} e^{-\mathrm{i}\,\mathbf{k}\,(\mathbf{y}-\mathbf{x})}\, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^n} \tag{1.26}$$

is one of the many presentations of the n-dimensional Dirac delta function.

Chapter 2

Classical field theory

2.1 Lagrangian formalism

The Lagrangian formalism for classical mechanics admits a straightforward generalization to classical field theory. To illustrate this formalism, we shall work on the d-dimensional Minkowski spacetime $M = (\mathbb{R}^d, \eta)$ and consider a field

$$\phi : M = \mathbb{R}^d \longrightarrow \mathcal{Q} = \mathbb{R}^k \ , \quad x \longmapsto \phi(x) \tag{2.1}$$

that takes values in a k-dimensional real vector space $\mathcal{Q} = \mathbb{R}^k$. Note that such ϕ is a multicomponent field, i.e.

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_k(x) \end{pmatrix} \in \mathbb{R}^k \quad , \tag{2.2}$$

and we shall use the index notation $\phi_a(x)$, for $a = 1, \ldots, k$, to denote the individual components.

An *action functional* for the field ϕ has the general form

$$S[\phi] = \int_{\mathbb{R}^d} \mathcal{L}(\phi, \partial\phi)\, \mathrm{d}x \quad , \tag{2.3}$$

where the integral is over the volume element

$$\mathrm{d}x \ := \ \mathrm{d}t\, \mathrm{d}x^1 \cdots \mathrm{d}x^{d-1} \tag{2.4}$$

of spacetime $\mathbb{R}^d$. The integrand $\mathcal{L}(\phi, \partial\phi)$ is called the *Lagrangian density* and it is assumed to be a real-valued local function, i.e. its value $\mathcal{L}(\phi, \partial\phi)(x)$ at the point $x \in \mathbb{R}^d$ is a function of $\phi(x)$ and of its partial derivatives $\partial_\mu \phi(x) := \frac{\partial \phi(x)}{\partial x^\mu}$ at the same point x. We use square brackets $S[\phi]$ to denote the argument of the action in order to emphasize that S is a functional, i.e. a function on the set

of functions/fields $\phi : \mathbb{R}^d \to \mathbb{R}^k$. The classical dynamics of the field ϕ is defined by extremizing the action

$$\delta S[\phi] = 0 \quad . \tag{2.5}$$

With a similar calculation as in the case of Lagrangian mechanics, one derives from this the *Euler-Lagrange equations*

$$\partial_\mu \Big(\frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial(\partial_\mu \phi_a)} \Big) - \frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial \phi_a} = 0 \quad , \tag{2.6}$$

for all $a = 1, \dots, k$. For your convenience, let me briefly repeat this calculation

$$\begin{aligned}
\delta S[\phi] &= \int_{\mathbb{R}^d} \Big(\frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial \phi_a}\, \delta\phi_a + \frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial(\partial_\mu \phi_a)}\, \delta(\partial_\mu \phi_a) \Big)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \Big(\frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial \phi_a}\, \delta\phi_a + \frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial(\partial_\mu \phi_a)}\, \partial_\mu(\delta\phi_a) \Big)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \Big(\frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial \phi_a}\, \delta\phi_a - \partial_\mu \Big(\frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial(\partial_\mu \phi_a)} \Big) \delta\phi_a \Big)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \Big(\frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial \phi_a} - \partial_\mu \Big(\frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial(\partial_\mu \phi_a)} \Big) \Big)\, \delta\phi_a\, \mathrm{d}x = 0 \quad ,
\end{aligned} \tag{2.7}$$

where step three uses integration by parts together with the usual hypothesis that the variation $\delta\phi_a$ vanishes at the boundary/infinity.

Example 2.1 (Real Klein-Gordon field)**.** Consider a real scalar field $\Phi(x) \in \mathbb{R}$ together with the quadratic action functional

$$S_{\mathrm{KG}}[\Phi] := \int_{\mathbb{R}^d} \Big(-\frac{1}{2}\, \eta^{\mu\nu}\, \partial_\mu \Phi\, \partial_\nu \Phi - \frac{m^2}{2}\, \Phi^2 \Big)\, \mathrm{d}x = \int_{\mathbb{R}^d} -\frac{1}{2} \left(\partial^\mu \Phi\, \partial_\mu \Phi + m^2\, \Phi^2 \right) \mathrm{d}x \quad , \tag{2.8}$$

where $\eta^{\mu\nu}$ denotes the inverse Minkowski metric and $m \geq 0$ is a parameter that will be interpreted later as the mass of a particle. The first summand is called the *kinetic term* and the second summand is called the *mass term*. The corresponding Euler-Lagrange equation (2.6) reads as

$$-\partial^2 \Phi + m^2\, \Phi \; := -\eta^{\mu\nu} \partial_\mu \partial_\nu \Phi + m^2\, \Phi = 0 \tag{2.9}$$

and it is called the *Klein-Gordon equation*. Note that (2.9) is a *linear* partial differential equation on the Minkowski spacetime, which is a consequence of the fact that the action functional (2.8) is quadratic in the field. One can introduce nonlinearities by adding higher-order terms to the action functional, e.g.

$$S_{\mathrm{KG+int}}[\Phi] := \int_{\mathbb{R}^d} \Big(-\frac{1}{2}\, \partial^\mu \Phi\, \partial_\mu \Phi - \frac{m^2}{2}\, \Phi^2 - V(\Phi) \Big)\, \mathrm{d}x \tag{2.10}$$

for a potential function $V(\Phi)$ that is a polynomial of degree ≥ 3 in Φ. The resulting Euler-Lagrange equation then reads as

$$-\partial^2 \Phi + m^2\, \Phi + V'(\Phi) = 0 \quad , \tag{2.11}$$

which is a nonlinear partial differential equation because the derivative $V'(\Phi)$ is by our hypothesis a polynomial of degree ≥ 2 in Φ.

Example 2.2 (Electromagnetic potential)**.** Consider a covector field $A_\mu(x)$ and define the antisymmetric $(0,2)$-tensor field

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \tag{2.12}$$

The physical interpretation of A_μ is that of the electromagnetic potential and $F_{\mu\nu}$ is the so-called field strength tensor. As a direct consequence of its definition, $F_{\mu\nu}$ satisfies the identity

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \quad . \tag{2.13}$$

Consider further the quadratic action functional

$$S_{\mathrm{MW}}[A] := \int_{\mathbb{R}^d} -\frac{1}{4}\,\eta^{\mu\rho}\,\eta^{\nu\sigma}\,F_{\mu\nu}\,F_{\rho\sigma}\,\mathrm{d}x = \int_{\mathbb{R}^d} -\frac{1}{4}\,F^{\mu\nu}\,F_{\mu\nu}\,\mathrm{d}x \tag{2.14}$$

whose Euler-Lagrange equations (2.6) read as

$$\partial_\mu F^{\mu\nu} = 0 \quad . \tag{2.15}$$

In spacetime dimension $d = 4$, the system of partial differential equations (2.13) and (2.15) is equivalent to Maxwell's equations in vacuum, namely $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, $\nabla \cdot \mathbf{E} = 0$ and $\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$. This example will be studied in more depth later in Chapter 6.

Example 2.3 (Complex-valued fields)**.** The Lagrangian formalism for field theory also works for complex-valued fields $\phi : \mathbb{R}^d \to \mathbb{C}^k$. Indeed, decomposing each component $\phi_a(x) =: \rho_a(x) + \mathrm{i}\,\chi_a(x)$ into its real part ρ_a and imaginary part χ_a allows us to regard ϕ as a $2k$-dimensional real field

$$\tilde{\phi} : \mathbb{R}^d \longrightarrow \mathbb{R}^{2k} \;, \quad x \longmapsto \tilde{\phi}(x) = \begin{pmatrix} \rho_1(x) \\ \vdots \\ \rho_k(x) \\ \chi_1(x) \\ \vdots \\ \chi_k(x) \end{pmatrix} \quad . \tag{2.16}$$

The formulas from this section then apply to $\tilde{\phi}$.

There is an equivalent, but simpler way to deal with complex-valued fields, which we are going to illustrate for a complex scalar field $\Phi(x) \in \mathbb{C}$ whose complex conjugate we denote by $\Phi^*(x) \in \mathbb{C}$. Since the action functional is by definition real-valued, one has to combine $\Phi(x)$ and $\Phi^*(x)$ in the Lagrangian density in order to obtain a real number. For example, the action functional

$$S_{\mathrm{KG}_{\mathbb{C}}+\mathrm{int}}[\Phi, \Phi^*] := \int_{\mathbb{R}^d} \Big(-\partial^\mu \Phi^*\,\partial_\mu \Phi - m^2\,\Phi^*\,\Phi - V(\Phi^*\,\Phi) \Big)\,\mathrm{d}x \tag{2.17}$$

achieves this by taking the absolute value of complex numbers. To obtain the Euler-Lagrange equations for this system, we treat Φ and Φ^* as independent fields. The Euler-Lagrange equation for Φ then reads as

$$\begin{aligned} 0 &= \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \Big) - \frac{\partial \mathcal{L}}{\partial \Phi} = \partial_\mu \Big(-\partial^\mu \Phi^* \Big) + m^2\,\Phi^* + V'(\Phi^*\,\Phi)\,\Phi^* \\ &= -\partial^2 \Phi^* + m^2\,\Phi^* + V'(\Phi^*\,\Phi)\,\Phi^* \end{aligned} \tag{2.18a}$$

and the one for Φ^* reads as

$$0 = \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}\Big) - \frac{\partial\mathcal{L}}{\partial\Phi^*} = \partial_\mu\Big(-\partial^\mu\Phi\Big) + m^2\,\Phi + V'(\Phi^*\,\Phi)\,\Phi$$
$$= -\partial^2\Phi + m^2\,\Phi + V'(\Phi^*\,\Phi)\,\Phi \quad . \tag{2.18b}$$

Observe that the two equations (2.18) are the complex conjugates of each other. They also agree with the Euler-Lagrange equations that one would obtain by decomposing $\Phi(x) = \rho(x) + \mathrm{i}\,\chi(x)$ into its real and imaginary part. (This is a good exercise for you!) For a trivial potential $V = 0$, one obtains the so-called complex Klein-Gordon action

$$S_{\mathrm{KG}_\mathbb{C}}[\Phi,\Phi^*] := \int_{\mathbb{R}^d} -\Big(\partial^\mu\Phi^*\,\partial_\mu\Phi + m^2\,\Phi^*\,\Phi\Big)\,\mathrm{d}x \quad , \tag{2.19}$$

whose Euler-Lagrange equations are the complex Klein-Gordon equations

$$-\partial^2\Phi + m^2\,\Phi = 0 \quad , \qquad -\partial^2\Phi^* + m^2\,\Phi^* = 0 \quad , \tag{2.20}$$

where again one follows from the other by complex conjugation.

2.2 Symmetries and Noether's theorem

Many classical field theories admit *symmetries*, by which one means transformations $T : \phi(x) \mapsto T\phi(x)$ on the set of fields that leave invariant the action functional, i.e.

$$S[T\phi] = S[\phi] \quad . \tag{2.21}$$

For instance, all examples presented in Section 2.1 are invariant under Poincaré transformations because their Lagrangian densities are constructed via the tensor calculus on Minkowski spacetime. Let us illustrate this in detail for the example of the real Klein-Gordon field from Example 2.1.

Example 2.4 (Poincaré symmetries of the real Klein-Gordon field)**.** Recall our conventions for Poincaré transformations $x'^\mu = \Lambda^\mu{}_\nu\,x^\nu + b^\mu$ from Section 1.4. A real scalar field transforms under Poincaré transformations as $\Phi'(x') = \Phi(x)$, i.e. the value of the transformed field Φ' at the transformed point x' is equal to the value of Φ at x. This point of view on Poincaré transformations is called *passive* since the field Φ does not get transformed to a new field, but it is only expressed in a different set of coordinates x'. The *active* point of view on Poincaré transformations consists of evaluating the transformed field Φ' at the old point x and interpreting this as a transformation $T : \Phi(x) \mapsto T\Phi(x) = \Phi'(x)$ on the set of fields. Inverting the Poincaré transformation $x' = \Lambda x + b$ gives $x = \Lambda^{-1}(x' - b)$, which when combined with the transformation law $\Phi'(x') = \Phi(x)$ yields

$$T : \Phi(x) \longmapsto T\Phi(x) = \Phi'(x) = \Phi\big(\Lambda^{-1}(x-b)\big) \quad . \tag{2.22}$$

In index notation, the argument of Φ reads as $(\Lambda^{-1}(x-b))^\nu = (x^\rho - b^\rho)\,\Lambda_\rho{}^\nu$. Using the chain rule for partial differentiation, we find that the partial derivatives of the transformed field $T\Phi$ are given

by

$$
\begin{aligned}
(\partial_\mu T\Phi)(x) &= \frac{\partial}{\partial x^\mu}\Phi(\Lambda^{-1}(x-b)) \\
&= \frac{\partial\big((x^\rho - b^\rho)\,\Lambda_\rho{}^\nu\big)}{\partial x^\mu}\;\; (\partial_\nu\Phi)(\Lambda^{-1}(x-b)) \\
&= \Lambda_\mu{}^\nu\,(\partial_\nu\Phi)(\Lambda^{-1}(x-b)) \quad , \qquad (2.23)
\end{aligned}
$$

which we recognize as the transformation law of a covector field under (active) Poincaré transformations. With these preparations, we can now confirm Poincaré invariance of the Klein-Gordon action (2.8) by a simple calculation

$$
\begin{aligned}
S_{\mathrm{KG}}[T\Phi] &= \int_{\mathbb{R}^d} -\frac{1}{2}\left(\eta^{\mu\nu}\,(\partial_\mu T\Phi)(x)\,(\partial_\nu T\Phi)(x) + m^2\,(T\Phi)^2(x)\right)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} -\frac{1}{2}\left(\eta^{\mu\nu}\,\Lambda_\mu{}^\rho\,\Lambda_\nu{}^\sigma\,(\partial_\rho\Phi)(\Lambda^{-1}(x-b))\,(\partial_\sigma\Phi)(\Lambda^{-1}(x-b)) + m^2\,\Phi^2(\Lambda^{-1}(x-b))\right)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} -\frac{1}{2}\left(\eta^{\rho\sigma}\,(\partial_\rho\Phi)(\Lambda^{-1}(x-b))\,(\partial_\sigma\Phi)(\Lambda^{-1}(x-b)) + m^2\,\Phi^2(\Lambda^{-1}(x-b))\right)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} -\frac{1}{2}\left(\eta^{\rho\sigma}\,(\partial_\rho\Phi)(y)\,(\partial_\sigma\Phi)(y) + m^2\,\Phi^2(y)\right)\mathrm{d}y = S_{\mathrm{KG}}[\Phi] \quad . \qquad (2.24)
\end{aligned}
$$

The third step uses the property (1.19) of the inverse Minkowski metric. In the fourth step we have changed the integration variables to $y := \Lambda^{-1}(x-b)$ and used the change of variables formula $\mathrm{d}y = |\det(\Lambda^{-1})|\,\mathrm{d}x = \mathrm{d}x$ for the volume element, together with the fact that $\det(\Lambda) = \pm 1$ which follows by taking determinants on both sides of (1.17). (Since as the default we consider proper and orthochronous Poincaré transformations, we even have $\det(\Lambda) = +1$.)

Poincaré symmetries are an example of so-called *spacetime symmetries*, which are by definition transformations $T : \phi(x) \mapsto T\phi(x)$ such that the value of the transformed field $T\phi(x)$ at the point x is determined from the value of ϕ at a different point. Compare this with (2.22), which shows that the Poincaré transformed scalar field $T\Phi$ at x is determined from the field Φ at $\Lambda^{-1}(x-b)$. Another type of symmetries is given by the so-called *internal symmetries* for which the transformed field $T\phi(x)$ at x is determined from the field $\phi(x)$ at the same point x. It is again best to illustrate this by an example.

Example 2.5 (Internal symmetries of the real and complex Klein-Gordon field)**.** For a real scalar field $\Phi(x) \in \mathbb{R}$, we can consider the transformation $T : \Phi(x) \mapsto T\Phi(x) = -\Phi(x)$ that flips the sign. This transformation is a symmetry of the Klein-Gordon action (2.8), as one can easily check

$$
\begin{aligned}
S_{\mathrm{KG}}[T\Phi] &= \int_{\mathbb{R}^d} -\frac{1}{2}\left(\eta^{\mu\nu}\,(\partial_\mu T\Phi)(x)\,(\partial_\nu T\Phi)(x) + m^2\,(T\Phi)^2(x)\right)\mathrm{d}x \\
&= (-1)^2\int_{\mathbb{R}^d} -\frac{1}{2}\left(\eta^{\mu\nu}\,(\partial_\mu\Phi)(x)\,(\partial_\nu\Phi)(x) + m^2\,\Phi^2(x)\right)\mathrm{d}x = S_{\mathrm{KG}}[\Phi] \quad . \qquad (2.25)
\end{aligned}
$$

Note that this transformation is also a symmetry of the action functional (2.10) with interaction terms, provided that the potential $V(\Phi)$ is an even function, i.e. $V(-\Phi) = V(\Phi)$. Since $T\Phi(x)$ at x is determined from $\Phi(x)$ at the same point x, this is an example of an internal symmetry.

The case of a complex scalar field $\Phi(x) \in \mathbb{C}$ is richer, in the sense that there is not only one but an infinite family of internal symmetries that is labeled by a continuous parameter. Consider the transformation $T : \Phi(x) \mapsto T\Phi(x) = e^{-\mathrm{i}\,\alpha}\,\Phi(x)$ that rotates the field Φ by any constant complex phase $e^{-\mathrm{i}\,\alpha} \in \mathsf{U}(1) \subseteq \mathbb{C}$, i.e. α is taken to be independent of x. The transformation of the complex conjugate field Φ^* is then given by $T : \Phi^*(x) \mapsto e^{\mathrm{i}\,\alpha}\,\Phi^*(x)$. The action (2.17) and its special case (2.19) are clearly invariant under this transformation, because every summand involves an equal power of Φ and Φ^*, such that the complex phases cancel out. For example, for the quadratic terms we have that $T\Phi^*(x)\,T\Phi(x) = e^{\mathrm{i}\,\alpha}\,\Phi^*(x)\,e^{-\mathrm{i}\,\alpha}\,\Phi(x) = \Phi^*(x)\,\Phi(x)$. Since there are infinitely many complex phases $e^{-\mathrm{i}\,\alpha} \in \mathsf{U}(1) \subseteq \mathbb{C}$, corresponding to choosing $\alpha \in [0, 2\pi)$, we found a continuous family of internal symmetries for the complex scalar field. It is worthwhile to note that constancy of the phase $e^{-\mathrm{i}\,\alpha}$ is crucial for having a symmetry: If $e^{-\mathrm{i}\,\alpha(x)}$ would depend on x, then the partial derivatives of the transformed field receive an extra term $(\partial_\mu T\Phi)(x) = \partial_\mu(e^{-\mathrm{i}\,\alpha(x)}\,\Phi(x)) = e^{-\mathrm{i}\,\alpha(x)}\,(\partial_\mu\Phi)(x) - \mathrm{i}\,(\partial_\mu\alpha)(x)\,e^{\mathrm{i}\,\alpha(x)}\,\Phi(x)$, which would spoil the invariance of the kinetic term in the actions (2.17) and (2.19). Such local, in the sense of x-dependent, internal transformations will play an important role later in this module when we will discuss *gauge theories*.

You might now ask: What's the point about having field theories with symmetries? What do I gain from this? These are indeed very good questions that can be answered in different ways. Let me present you two particularly important answers.

1. Field theories that admit Poincaré symmetries are compatible with the laws of special relativity. This means that they will be particularly useful for applications to high-energy physics, which involves very fast particles and hence is the natural habitat of special relativity.

2. To each continuous family of symmetries there is an associated *conserved current*, which in particular yields a time-independent quantity called the conserved charge. This holds true independently of whether one has internal or spacetime symmetries, and it is know as *Noether's theorem*.

Our goal for the rest of this section is to derive Noether's theorem and to provide examples of conserved charges. The terminology *continuous family of symmetries* means that we have a whole family of transformations $T_\beta : \phi(x) \mapsto T_\beta\phi(x)$, labeled by one or many continuous parameters $\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{R}^N$, that leaves invariant the action $S[T_\beta\phi] = S[\phi]$. The Poincaré transformations from Example 2.4 are an example, where the continuous parameters label the different rotation angles, Lorentz boosts and translations of a Poincaré transformation, and also the phase rotations for a complex scalar field from Example 2.5 are an example, where the parameter labels the complex phase. On the other hand, the sign transformation $\Phi(x) \mapsto -\Phi(x)$ for a real scalar field is *not* a continuous family of symmetries, because there is no parameter we can vary.

Assuming that the transformation $T_0\phi(x) = \phi(x)$ corresponding to the parameter $\beta = 0$ is the identity transformation and that $T_\beta\phi(x)$ is differentiable at $\beta = 0$, we can Taylor expand

$$T_\beta\phi(x) = \phi(x) + \delta_\beta\phi(x) + \mathcal{O}(\beta^2) \tag{2.26}$$

to first order in β to obtain the infinitesimal transformation $\delta_\beta\phi$ of ϕ under the family of transformations T_β. Expanding also the invariance condition $S[T_\beta\phi] = S[\phi]$ of the action functional to first order in β and considering the integrands, one obtains that the infinitesimal transformation of the

Lagrangian density

$$\delta_\beta \mathcal{L}(\phi, \partial\phi) := \frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial \phi_a} (\delta_\beta \phi)_a + \frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial(\partial_\mu \phi_a)} \partial_\mu (\delta_\beta \phi)_a = \partial_\mu F^\mu \tag{2.27}$$

must be the divergence of some vector field $F^\mu(x)$. This now allows us to formulate and prove Noether's theorem.

Theorem 2.6 (Noether's theorem)**.** *Let $\delta_\beta\phi$ be an infinitesimal symmetry transformation, e.g. obtained by Taylor expanding a continuous family of symmetries (2.26). Then the vector field*

$$J^\mu := \frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial(\partial_\mu \phi_a)} (\delta_\beta \phi)_a - F^\mu \tag{2.28}$$

is a conserved current in the sense that its divergence

$$\partial_\mu J^\mu = 0 \tag{2.29}$$

vanishes whenever the field ϕ satisfies the Euler-Lagrange equations (2.6).

Proof. We compute

$$\begin{aligned}
\partial_\mu J^\mu &= \partial_\mu \left(\frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial(\partial_\mu \phi_a)} (\delta_\beta \phi)_a \right) - \partial_\mu F^\mu \\
&= \partial_\mu \left(\frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial(\partial_\mu \phi_a)} \right) (\delta_\beta \phi)_a + \frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial(\partial_\mu \phi_a)} \partial_\mu (\delta_\beta \phi)_a - \frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial \phi_a} (\delta_\beta \phi)_a - \frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial(\partial_\mu \phi_a)} \partial_\mu (\delta_\beta \phi)_a \\
&= \left(\partial_\mu \left(\frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial(\partial_\mu \phi_a)} \right) - \frac{\partial \mathcal{L}(\phi, \partial\phi)}{\partial \phi_a} \right) (\delta_\beta \phi)_a = 0 \quad .
\end{aligned} \tag{2.30}$$

In the second step we have used the Leibniz/product rule for ∂_μ and inserted (2.27). In the third step we have used that the 2nd and the 4th term cancel each other. The last step follows by using the Euler-Lagrange equations (2.6). □

Remark 2.7. The requirement that ϕ satisfies the Euler-Lagrange equations (2.6) is in general necessary for the Noether current J^μ to be conserved. Satisfying the Euler-Lagrange equations is often called *on-shell* in physics slang, which is why you will probably hear the terminology that J^μ is an on-shell conserved current.

Remark 2.8. The reason why a conserved current J^μ is so useful is that it has an associated conserved charge, which is defined by the integral

$$Q(t) := \int_{\mathbb{R}^{d-1}} J^0(t, \mathbf{x}) \, \mathrm{d}\mathbf{x} \tag{2.31}$$

of the 0-component of J^μ over the space coordinates $\mathbf{x} \in \mathbb{R}^{d-1}$. The charge is time-independent, hence the name conserved, as shown by the following calculation

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = \int_{\mathbb{R}^{d-1}} \frac{\partial J^0(t, \mathbf{x})}{\partial t} \, \mathrm{d}\mathbf{x} = - \int_{\mathbb{R}^{d-1}} \nabla \cdot \mathbf{J}(t, \mathbf{x}) \, \mathrm{d}\mathbf{x} = 0 \quad , \tag{2.32}$$

where in the second step we have used current conservation $0 = \partial_\mu J^\mu = \frac{\partial J^0}{\partial t} + \nabla \cdot \mathbf{J}$ and in the third step Gauss' integration theorem. (As usual, it is assumed that the fields fall off sufficiently fast at infinity so that there are no boundary terms.)

Example 2.9 (Relativistic momentum of the real Klein-Gordon field)**.** Recall from Example 2.4 that translations of spacetime define a continuous family of symmetries $T_b : \Phi(x) \mapsto T_b\Phi(x) = \Phi(x-b)$ for the real Klein-Gordon field. Taking the first order Taylor expansion as in (2.26) gives

$$\delta_b\Phi = -b^\mu\,\partial_\mu\Phi \quad . \tag{2.33}$$

For the transformation of the Lagrangian density (2.27) one finds

$$\delta_b\mathcal{L} = \partial_\mu\big(-b^\mu\,\mathcal{L}\big) \quad \Longrightarrow \quad F^\mu = -b^\mu\,\mathcal{L} \quad . \tag{2.34}$$

The associated Noether current from Theorem 2.6 then reads as

$$\begin{aligned} J^\mu &= \partial^\mu\Phi\, b^\nu\,\partial_\nu\Phi + b^\mu\mathcal{L} \\ &= b_\nu\left(\partial^\mu\Phi\,\partial^\nu\Phi - \frac{1}{2}\,\eta^{\mu\nu}\Big(\partial^\rho\Phi\,\partial_\rho\Phi + m^2\,\Phi^2\Big)\right) \quad . \end{aligned} \tag{2.35}$$

The term in the parenthesis is called the *energy-momentum tensor* and it is often denoted by

$$T^{\mu\nu} := \partial^\mu\Phi\,\partial^\nu\Phi - \frac{1}{2}\,\eta^{\mu\nu}\Big(\partial^\rho\Phi\,\partial_\rho\Phi + m^2\,\Phi^2\Big) \quad . \tag{2.36}$$

Note that current conservation $\partial_\mu J^\mu = 0$, for all translation vectors b^ν, is equivalent to the conservation law $\partial_\mu T^{\mu\nu} = 0$ of the energy-momentum tensor that you have probably seen before in your relativity module. The conserved charges associated with the energy-momentum tensor define the relativistic momentum

$$P^\mu := \int_{\mathbb{R}^{d-1}} T^{0\mu}\,\mathrm{d}\mathbf{x} \tag{2.37a}$$

of the Klein-Gordon field. Explicitly, the $\mu = 0$-component reads as

$$E := P^0 = \int_{\mathbb{R}^{d-1}} T^{00}\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^{d-1}} \left(\frac{1}{2}\dot\Phi^2 + \frac{1}{2}(\nabla\Phi)^2 + \frac{m^2}{2}\,\Phi^2\right)\mathrm{d}\mathbf{x} \quad , \tag{2.37b}$$

where $\dot\Phi := \frac{\partial\Phi}{\partial t}$ denotes the time derivative, and it describes the energy. (As we will see in Section 2.3, this agrees with the Hamiltonian.) The spatial components

$$P^i = -\int_{\mathbb{R}^{d-1}} \dot\Phi\,\partial^i\Phi\,\mathrm{d}\mathbf{x} \quad , \tag{2.37c}$$

for $i = 1, \dots, d-1$, describe the momentum.

Example 2.10 (Relativistic angular momentum of the real Klein-Gordon field)**.** Recalling again Example 2.4, we consider now the case of a Lorentz transformation $x' = \Lambda x$. To obtain a suitable continuous family of Lorentz transformations, we write $\Lambda = e^\omega = \sum_{n=0}^\infty \frac{\omega^n}{n!}$ as the exponential of a matrix ω. Expanding the condition (1.17) to first order in ω yields

$$\begin{aligned} \eta_{\mu\nu}\,\Lambda^\mu{}_\rho\,\Lambda^\nu{}_\sigma &= \eta_{\mu\nu}\,(\delta^\mu{}_\rho + \omega^\mu{}_\rho)\,(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma) + \mathcal{O}(\omega^2) \\ &= \eta_{\rho\sigma} + \omega_{\rho\sigma} + \omega_{\sigma\rho} + \mathcal{O}(\omega^2) = \eta_{\rho\sigma} \quad , \end{aligned} \tag{2.38}$$

hence the index lowering $\omega_{\mu\nu} := \eta_{\mu\rho}\,\omega^{\rho}{}_{\nu}$ must be antisymmetric

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \quad . \tag{2.39}$$

One can check that the latter condition implies that $\Lambda = e^{\omega}$ satisfies (1.17) to all orders in ω. Hence, we obtain a continuous family of symmetries $T_\omega : \Phi(x) \mapsto T_\omega\Phi(x) = \Phi(e^{-\omega}x)$ for the real Klein-Gordon field. Taking the first order Taylor expansion as in (2.26) gives

$$\delta_\omega\Phi = -\omega^{\mu}{}_{\nu}\,x^\nu\,\partial_\mu\Phi \quad . \tag{2.40}$$

For the transformation of the Lagrangian density (2.27) one finds

$$\delta_\omega\mathcal{L} = -\omega^{\mu}{}_{\nu}\,x^\nu\,\partial_\mu\mathcal{L} = \partial_\mu\Big(-\omega^{\mu}{}_{\nu}\,x^\nu\,\mathcal{L}\Big) \quad\Longrightarrow\quad F^\mu = -\omega^{\mu}{}_{\nu}\,x^\nu\,\mathcal{L} \quad , \tag{2.41}$$

where in the second equality we have used that $\omega^{\mu}{}_{\mu} = \eta^{\mu\nu}\omega_{\nu\mu} = 0$ as a consequence of the antisymmetry condition (2.39). The associated Noether current from Theorem 2.6 then reads as

$$J^\mu = -\omega_{\nu\rho}\,x^\nu\,T^{\mu\rho} = \frac{1}{2}\,\omega_{\nu\rho}\Big(x^\rho\,T^{\mu\nu} - x^\nu\,T^{\mu\rho}\Big) \quad , \tag{2.42}$$

where $T^{\mu\rho}$ denotes the energy-momentum tensor (2.36) and in the last step we used that $\omega_{\nu\rho} = -\omega_{\rho\nu}$ is antisymmetric. Denoting the term in the parenthesis by

$$(\mathcal{J}^\mu)^{\rho\nu} := x^\rho\,T^{\mu\nu} - x^\nu\,T^{\mu\rho} \quad , \tag{2.43}$$

current conservation $\partial_\mu J^\mu = 0$, for all parameters $\omega_{\nu\rho}$, is equivalent to the conservation law $\partial_\mu(\mathcal{J}^\mu)^{\rho\nu} = 0$. The conserved charges associated with $(\mathcal{J}^\mu)^{\rho\nu}$ define the relativistic angular momentum

$$L^{\rho\nu} := \int_{\mathbb{R}^{d-1}} (\mathcal{J}^0)^{\rho\nu}\;\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^{d-1}} \Big(x^\rho\,T^{0\nu} - x^\nu\,T^{0\rho}\Big)\;\mathrm{d}\mathbf{x} \tag{2.44}$$

of the Klein-Gordon field. Since the angular momentum (2.44) contains only contributions arising from the dynamics of the field, we can conclude that the Klein-Gordon field does not carry any internal angular momentum, a.k.a. spin.

Example 2.11 (Electric charge of the complex Klein-Gordon field)**.** Recall from Example 2.5 that the complex Klein-Gordon field admits a continuous family of internal symmetries $T_\alpha : \Phi(x) \mapsto T_\alpha\Phi(x) = e^{-\mathrm{i}\,\alpha}\,\Phi(x)$. Taking the first order Taylor expansion as in (2.26) gives

$$\delta_\alpha\Phi = -\mathrm{i}\,\alpha\,\Phi \quad , \qquad \delta_\alpha\Phi^* = \mathrm{i}\,\alpha\,\Phi^* \quad . \tag{2.45}$$

For the transformation of the Lagrangian density (2.27) one finds

$$\delta_\alpha\mathcal{L} = 0 \quad\Longrightarrow\quad F^\mu = 0 \tag{2.46}$$

because the Lagrangian density (2.19) is invariant under phase rotations. The associated Noether current from Theorem 2.6 then reads as

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\,\delta_\alpha\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^*)}\,\delta_\alpha\Phi^* = \alpha\Big(\mathrm{i}\,\partial^\mu\Phi^*\,\Phi - \mathrm{i}\,\partial^\mu\Phi\,\Phi^*\Big) \quad . \tag{2.47}$$

The term in the parenthesis is called the electromagnetic current of the complex Klein-Gordon field and it is often denoted by

$$j^\mu := \mathrm{i}\left(\Phi\,\partial^\mu\Phi^* - \Phi^*\,\partial^\mu\Phi\right) \quad . \tag{2.48}$$

The conserved charge associated with j^μ defines the electric charge

$$Q := \mathrm{i}\int_{\mathbb{R}^{d-1}}\left(\Phi^*\,\dot\Phi - \Phi\,\dot\Phi^*\right)\mathrm{d}\mathbf{x} \quad , \tag{2.49}$$

where $\dot\Phi = \frac{\partial\Phi}{\partial t} = \partial_0\Phi = -\partial^0\Phi$ denotes the time derivative. (The minus sign in the last step is due to $\eta^{00} = -1$ in the Minkowski metric.)

2.3 Hamiltonian formalism

The passage from the Lagrangian formalism to the Hamiltonian formalism in classical field theory is analogous to the case of classical mechanics. For your convenience, let me briefly recall the relevant steps.

Consider as in Section 2.1 a multicomponent field $\phi_a(x)$ on the d-dimensional Minkowski spacetime $(\mathbb{R}^d, \eta)$ with Lagrangian density $\mathcal{L}(\phi, \partial\phi)$. The *canonical momentum* associated with ϕ_a is defined by

$$\pi^a := \frac{\partial\mathcal{L}(\phi,\dot\phi,\nabla\phi)}{\partial\dot\phi_a} \quad , \tag{2.50}$$

where in the argument of $\mathcal{L}$ we have splitted the partial derivatives $\partial\phi$ into the time derivative $\dot\phi$ and the space derivatives $\nabla\phi$. Assuming that this equation can be solved for $\dot\phi_a = \dot\phi_a(\pi,\phi,\nabla\phi)$ as a function of the π's, the ϕ's and the $\nabla\phi$'s, we define the *Hamiltonian density* by

$$\mathcal{H}(\pi,\phi,\nabla\phi) := \pi^a\,\dot\phi_a(\pi,\phi,\nabla\phi) - \mathcal{L}(\phi,\dot\phi(\pi,\phi,\nabla\phi),\nabla\phi) \quad . \tag{2.51}$$

Note that the Hamiltonian density is a local function of π, ϕ and $\nabla\phi$, i.e. its value at the point x is determined by $\pi(x)$, $\phi(x)$ and $\nabla\phi(x)$ at the same point x. The *Hamiltonian* is then defined by integrating

$$H[\pi,\phi] := \int_{\mathbb{R}^{d-1}}\mathcal{H}(\pi,\phi,\nabla\phi)\,\mathrm{d}\mathbf{x} \tag{2.52}$$

the Hamiltonian density over space $\mathbf{x}\in\mathbb{R}^{d-1}$, where we recall that the square bracket notation $H[\pi,\phi]$ is used to emphasize that H is a functional. To obtain Hamilton's equations in field theory, we have to take *functional derivatives* of the Hamiltonian. These are defined as follows: In analogy to varying an action functional, we consider

$$\begin{aligned}\delta H[\pi,\phi] &= \int_{\mathbb{R}^{d-1}}\left(\frac{\partial\mathcal{H}(\pi,\phi,\nabla\phi)}{\partial\pi^a}\,\delta\pi^a + \frac{\partial\mathcal{H}(\pi,\phi,\nabla\phi)}{\partial\phi_a}\,\delta\phi_a + \frac{\partial\mathcal{H}(\pi,\phi,\nabla\phi)}{\partial(\partial_i\phi_a)}\,\partial_i(\delta\phi_a)\right)\mathrm{d}\mathbf{x}\\ &= \int_{\mathbb{R}^{d-1}}\left(\frac{\partial\mathcal{H}(\pi,\phi,\nabla\phi)}{\partial\pi^a}\,\delta\pi^a + \left(\frac{\partial\mathcal{H}(\pi,\phi,\nabla\phi)}{\partial\phi_a} - \partial_i\left(\frac{\partial\mathcal{H}(\pi,\phi,\nabla\phi)}{\partial(\partial_i\phi_a)}\right)\right)\delta\phi_a\right)\mathrm{d}\mathbf{x} \quad .\end{aligned} \tag{2.53}$$

The functional derivative along π^a is then the coefficient in front of $\delta\pi^a$ and the functional derivative along ϕ_a is the coefficient in front of $\delta\phi_a$. The following notations are often used

$$\frac{\delta H[\pi,\phi]}{\delta \pi^a} = \frac{\partial \mathcal{H}(\pi,\phi,\nabla\phi)}{\partial \pi^a} \quad , \qquad \frac{\delta H[\pi,\phi]}{\delta \phi_a} = \frac{\partial \mathcal{H}(\pi,\phi,\nabla\phi)}{\partial \phi_a} - \partial_i\Big(\frac{\partial \mathcal{H}(\pi,\phi,\nabla\phi)}{\partial(\partial_i\phi_a)}\Big) \quad . \tag{2.54}$$

Hamilton's equations in field theory then read as

$$\dot{\phi}_a = \frac{\delta H[\pi,\phi]}{\delta \pi^a} \quad , \qquad \dot{\pi}^a = -\frac{\delta H[\pi,\phi]}{\delta \phi_a} \quad . \tag{2.55}$$

Remark 2.12. The Hamiltonian formalism for field theory has both advantages and disadvantages in comparison to the Lagrangian formalism. Its main advantage is that it serves as a bridge to QFT, since canonical quantization is typically described starting from the Hamiltonian formalism. Its main disadvantage is that Poincaré symmetry is not manifest in this approach. In fact, in order to derive Hamilton's equations (2.55), we had to make an arbitrary split between time and space $x = (t, \mathbf{x})$ and treat these concepts very differently. For instance, the canonical momenta (2.50) are determined from the time derivatives $\dot{\phi}_a$, the Hamiltonian (2.52) involves an integration over only the space coordinates $\mathbf{x}$, and Hamilton's equations (2.55) involve taking only time derivatives. Since Poincaré transformations do in general mix between time and space coordinates, these quantities are not manifestly Poincaré invariant. It therefore requires additional investigations whether or not a field theory that is formulated in the Hamiltonian formalism admits Poincaré symmetry. We will later carry out these investigations in the context of QFT.

Example 2.13 (Hamiltonian for the real Klein-Gordon field)**.** Consider the real Klein-Gordon field from Example 2.1. Splitting the derivatives in the action functional (2.8) into time and space components, we can write

$$S_{\mathrm{KG}}[\Phi] = \int_{\mathbb{R}^d} \frac{1}{2}\Big(\dot{\Phi}^2 - (\nabla\Phi)^2 - m^2\,\Phi^2\Big)\,\mathrm{d}x \quad . \tag{2.56}$$

For the canonical momentum (2.50) we find

$$\Pi = \dot{\Phi} \tag{2.57}$$

and the Hamiltonian (2.52) is given by

$$H[\Pi,\Phi] = \int_{\mathbb{R}^{d-1}} \frac{1}{2}\Big(\Pi^2 + (\nabla\Phi)^2 + m^2\,\Phi^2\Big)\,\mathrm{d}\mathbf{x} \quad . \tag{2.58}$$

Hamilton's equations (2.55) for the real Klein-Gordon field then read as

$$\dot{\Phi} = \Pi \quad , \qquad \dot{\Pi} = -m^2\,\Phi + \nabla^2\Phi \quad . \tag{2.59}$$

This is the first-order form of the Klein-Gordon equation $-\partial^2\Phi + m^2\,\Phi = 0$, as can be seen by inserting the first into the second equation and using that $\partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu = -\frac{\partial^2}{\partial t^2} + \nabla^2$.

It would be a good exercise for you to derive the Hamiltonian for the complex Klein-Gordon field from Example 2.3.

Further reading

For more details about classical field theories, see e.g. Nastase (Chapter 1), Greiner/Reinhardt (Chapter 2), Maggiore (Chapter 3) and Schwartz (Chapter 3) from our reading list in Section 1.3.

Chapter 3

Free quantum Klein-Gordon field

This chapter studies the canonical quantization of the free real Klein-Gordon field from Examples 2.1 and 2.13. It will also be illustrated in which sense a QFT has an associated theory of particles.

3.1 Field operator and dynamics

Recall that, in quantum mechanics, canonical quantization amounts to replacing the canonical coordinates q_a and p^a of the Hamiltonian formalism by operators that satisfy the canonical commutation relations (CCRs) from (1.8). It is important to emphasize that these commutation relations are formulated at a fixed instance of time $t_0 \in \mathbb{R}$, so when being more pedantic with notation one should write $q_a := q_a(t_0)$ and $p^a := p^a(t_0)$. We will mostly choose $t_0 = 0$, which simplifies our formulas as it avoids expressions of the form $t - t_0$.

The analog of the CCRs for a real Klein-Gordon field in the Hamiltonian formalism from Example 2.13 is given by considering the field $\Phi(\mathbf{x}) := \Phi(t_0, \mathbf{x})$ and its canonical momentum $\Pi(\mathbf{x}) := \Pi(t_0, \mathbf{x})$ at a fixed instance of time $t_0 \in \mathbb{R}$, which we suppress from the notation, and demand the commutation relations

$$\big[\Phi(\mathbf{x}), \Phi(\mathbf{y})\big] = 0 = \big[\Pi(\mathbf{x}), \Pi(\mathbf{y})\big] \quad , \qquad \big[\Phi(\mathbf{x}), \Pi(\mathbf{y})\big] = \mathrm{i}\,\delta(\mathbf{x} - \mathbf{y}) \quad , \tag{3.1}$$

where $\delta(\mathbf{x} - \mathbf{y})$ denotes the Dirac delta function. Because all operators are considered at the same value of time $t_0 \in \mathbb{R}$, but importantly not necessarily at the same positions in space $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{d-1}$, one often calls (3.1) the *equal-time commutation relations*. Comparison between (3.1) and (1.8) shows that, in the context of field theory, the index $a = 1, \ldots, N$ on q_a and p^a that counts the degrees of freedom is generalized to a continuous index $\mathbf{x} \in \mathbb{R}^{d-1}$ that corresponds to the different points in space. This is why people often say that QFT deals with infinitely many degrees of freedom, at least one for each point $\mathbf{x} \in \mathbb{R}^{d-1}$ in space. Note that the equal-time commutation relations are *local in space*, by which one means that any two operators corresponding to different points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{d-1}$ commute with each other. (Informally speaking, the Dirac delta function $\delta(\mathbf{x} - \mathbf{y})$ is 0 for $\mathbf{x} \neq \mathbf{y}$.)

The (too naive, as explained below) Hamiltonian operator for the free real Klein-Gordon field

is given by promoting the classical Hamiltonian (2.13) to an operator

$$H = \int_{\mathbb{R}^{d-1}} \frac{1}{2} \left(\Pi^2 + (\nabla \Phi)^2 + m^2 \, \Phi^2 \right) \mathrm{d}\mathbf{x} \quad , \tag{3.2}$$

where it is important to emphasize that Π and Φ are now operators in contrast to the classical fields in (2.13). Due to the spatial derivatives ∇ on Φ, the Hamiltonian mixes between the field operators $\Phi(\mathbf{x})$ corresponding to different points $\mathbf{x} \in \mathbb{R}^{d-1}$ in space. (Recall that the derivative of a function $f(x)$ is defined by $f'(x) = \lim_{\epsilon \to 0} (f(x+\epsilon) - f(x))/\epsilon$, hence it depends on f at the different, albeit infinitesimally close, points x and $x + \epsilon$.) Such interplay complicates finding solutions of Heisenberg's equation for the time evolution of operators, so thinking ahead we should do something about it. There is a simple way to get rid of this mixing coming from the spatial derivatives ∇, which is given by going from position space $\mathbf{x} \in \mathbb{R}^{d-1}$ to Fourier space $\mathbf{k} \in \mathbb{R}^{d-1}$ via a Fourier transform: Indeed, from the Fourier transformation formulas in Section 1.4, one easily sees that a derivative $\nabla f(\mathbf{x})$ in position space becomes a multiplication $\mathrm{i}\,\mathbf{k}\,\widetilde{f}(\mathbf{k})$ in Fourier space. Writing the operators

$$\Phi(\mathbf{x}) = \int_{\mathbb{R}^{d-1}} \widetilde{\Phi}(\mathbf{k}) \, e^{\mathrm{i}\,\mathbf{k}\,\mathbf{x}} \, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \quad \Pi(\mathbf{x}) = \int_{\mathbb{R}^{d-1}} \widetilde{\Pi}(\mathbf{k}) \, e^{\mathrm{i}\,\mathbf{k}\,\mathbf{x}} \, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \tag{3.3}$$

in Fourier space, one finds for the Hamiltonian operator (3.2)

$$\begin{aligned} H &= \int_{\mathbb{R}^{d-1}} \frac{1}{2} \left(\widetilde{\Pi}(\mathbf{k}) \, \widetilde{\Pi}(-\mathbf{k}) + (\mathbf{k}^2 + m^2) \; \widetilde{\Phi}(\mathbf{k}) \, \widetilde{\Phi}(-\mathbf{k}) \right) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \\ &= \int_{\mathbb{R}^{d-1}} \frac{1}{2} \left(\widetilde{\Pi}(\mathbf{k}) \, \widetilde{\Pi}(-\mathbf{k}) + \omega_{\mathbf{k}}^2 \; \widetilde{\Phi}(\mathbf{k}) \, \widetilde{\Phi}(-\mathbf{k}) \right) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \end{aligned} \tag{3.4}$$

where in the second line we have identified the relativistic energy

$$\omega_{\mathbf{k}} := \sqrt{\mathbf{k}^2 + m^2} \quad . \tag{3.5}$$

To carry out this calculation, which you should definitely do, one has to use the identity

$$\int_{\mathbb{R}^{d-1}} e^{\mathrm{i}\,(\mathbf{k}+\mathbf{q})\,\mathbf{x}} \, \mathrm{d}\mathbf{x} = (2\pi)^{d-1} \, \delta(\mathbf{k} + \mathbf{q}) \tag{3.6}$$

for the $d-1$-dimensional Dirac delta function. The Fourier transforms of the equal-time commutation relations (3.1) read as

$$[\widetilde{\Phi}(\mathbf{k}), \widetilde{\Phi}(\mathbf{q})] = 0 = [\widetilde{\Pi}(\mathbf{k}), \widetilde{\Pi}(\mathbf{q})] \quad , \qquad [\widetilde{\Phi}(\mathbf{k}), \widetilde{\Pi}(\mathbf{q})] = \mathrm{i}\,(2\pi)^{d-1} \, \delta(\mathbf{k} + \mathbf{q}) \quad , \tag{3.7}$$

as one can easily check using the formulas from Section 1.4.

A further simplification can be achieved by introducing, in analogy to the quantum harmonic oscillator from quantum mechanics, *annihilation operators* $a(\mathbf{k})$ and their adjoints the *creation operators* $a^\dagger(\mathbf{k}) := (a(\mathbf{k}))^\dagger$, for all $\mathbf{k} \in \mathbb{R}^{d-1}$. Because of the nonstandard Hermiticity conditions $(\widetilde{\Phi}(\mathbf{k}))^\dagger = \widetilde{\Phi}(-\mathbf{k})$ and $(\widetilde{\Pi}(\mathbf{k}))^\dagger = \widetilde{\Pi}(-\mathbf{k})$ in Fourier space, which follow from the usual Hermiticity conditions $(\Phi(\mathbf{x}))^\dagger = \Phi(\mathbf{x})$ and $(\Pi(\mathbf{x}))^\dagger = \Pi(\mathbf{x})$ in position space and the fact that taking adjoints changes the sign in the complex exponentials in (3.3), the annihilation and creation operators are

defined slightly differently than in quantum mechanics in order to ensure that they are adjoints of each other. In short, the expressions

$$\widetilde{\Phi}(\mathbf{k}) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \Big(a(\mathbf{k}) + a^\dagger(-\mathbf{k}) \Big) \quad , \qquad \widetilde{\Pi}(\mathbf{k}) = -\mathrm{i} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \Big(a(\mathbf{k}) - a^\dagger(-\mathbf{k}) \Big) \tag{3.8}$$

are compatible with the required Hermiticity conditions (note the $-\mathbf{k}$ in the argument of $a^\dagger$), and when inverted give

$$a(\mathbf{k}) = \sqrt{\frac{\omega_{\mathbf{k}}}{2}}\, \widetilde{\Phi}(\mathbf{k}) + \frac{\mathrm{i}}{\sqrt{2\omega_{\mathbf{k}}}}\, \widetilde{\Pi}(\mathbf{k}) \quad , \qquad a^\dagger(\mathbf{k}) = \sqrt{\frac{\omega_{\mathbf{k}}}{2}}\, \widetilde{\Phi}(-\mathbf{k}) - \frac{\mathrm{i}}{\sqrt{2\omega_{\mathbf{k}}}}\, \widetilde{\Pi}(-\mathbf{k}) \quad . \tag{3.9}$$

From (3.7) one then obtains the commutation relations

$$\big[a(\mathbf{k}), a(\mathbf{q})\big] = 0 = \big[a^\dagger(\mathbf{k}), a^\dagger(\mathbf{q})\big] \quad , \qquad \big[a(\mathbf{k}), a^\dagger(\mathbf{q})\big] = (2\pi)^{d-1}\, \delta(\mathbf{k}-\mathbf{q}) \tag{3.10}$$

for the annihilation and creation operators. Furthermore, inserting (3.8) into the Hamiltonian operator (3.4), one finds after a calculation (that you should do!) the expression

$$H = \int_{\mathbb{R}^{d-1}} \frac{\omega_{\mathbf{k}}}{2} \Big(a^\dagger(\mathbf{k})\, a(\mathbf{k}) + a(\mathbf{k})\, a^\dagger(\mathbf{k}) \Big) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \tag{3.11}$$

in terms of the annihilation and creation operators.

This form of the Hamiltonian shows that the free quantum Klein-Gordon field in Fourier space is simply an infinite family of quantum harmonic oscillators, one for each $\mathbf{k} \in \mathbb{R}^{d-1}$. However, this infinite number of degrees of freedom bites back! If we want to rewrite the Hamiltonian in the standard way such that the $a^\dagger$'s are to the left of the a's, we find using the commutation relations (3.10) that

$$H = \int_{\mathbb{R}^{d-1}} \omega_{\mathbf{k}}\, a^\dagger(\mathbf{k})\, a(\mathbf{k})\, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} + (2\pi)^{d-1}\, \delta(\mathbf{0}) \int_{\mathbb{R}^{d-1}} \frac{\omega_{\mathbf{k}}}{2}\, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \tag{3.12}$$

which is a nonsensible expression for two very different reasons: First, the Dirac delta function $(2\pi)^{d-1}\, \delta(\mathbf{0}) = \int_{\mathbb{R}^{d-1}} \mathrm{d}\mathbf{x}$ at zero is mathematically ill-defined. Note that this divergence results from the infinite volume of position space, i.e. large distances, which is why it is called an *infrared divergence*. Second, the integral $\int_{\mathbb{R}^{d-1}} \frac{\omega_{\mathbf{k}}}{2}\, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}}$ diverges because $\omega_{\mathbf{k}}$ grows as $|\mathbf{k}|$ for large $\mathbf{k}$. Note that this divergence results from large momenta, i.e. short distances in position space, which is why it is called an *ultraviolet divergence*. The way how to deal with such types of divergences is to establish a better operator ordering.

Definition 3.1 (Normal ordering)**.** Let A be an operator that is a polynomial in the annihilation and creation operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$. We define the *normal ordering* of A to be the operator $:A:$ that is obtained by placing in each summand all $a^\dagger(\mathbf{k})$ to the left of the $a(\mathbf{k})$.

Example 3.2. The normal ordering of the Hamiltonian is given by

$$:H: \; = \int_{\mathbb{R}^{d-1}} \omega_{\mathbf{k}}\, a^\dagger(\mathbf{k})\, a(\mathbf{k})\, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad . \tag{3.13}$$

This means that normal ordering subtracts the divergent term that we have identified in (3.12).

Let us now derive the *Heisenberg picture field operator* $\Phi(x) := \Phi(t, \mathbf{x})$, which is an operator on spacetime $x = (t, \mathbf{x}) \in \mathbb{R}^d$ (in contrast to only space $\mathbf{x} \in \mathbb{R}^{d-1}$) that is determined by solving Heisenberg's equation

$$\frac{\partial}{\partial t}\Phi(x) = \mathrm{i}\,[:H:, \Phi(x)] \quad , \tag{3.14}$$

subject to the initial condition $\Phi(0, \mathbf{x}) = \Phi(\mathbf{x})$. (We choose the initial time $t_0 = 0$ to be zero in order to avoid cluttering our formulas with a lot of t_0's. Of course, you are free to take any initial time $t_0 \in \mathbb{R}$ that you like.) Recall that the solution of this equation is given exponentiation

$$\Phi(x) = e^{\mathrm{i}\,t\,[:H:,-]}\,\Phi(\mathbf{x}) = e^{\mathrm{i}\,t\,:H:}\,\Phi(\mathbf{x})\,e^{-\mathrm{i}\,t\,:H:} \quad . \tag{3.15}$$

Combining (3.3) and (3.8), we can write the initial condition as

$$\begin{aligned}\Phi(\mathbf{x}) &= \int_{\mathbb{R}^{d-1}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a(\mathbf{k})\, e^{\mathrm{i}\,\mathbf{k}\,\mathbf{x}} + a^\dagger(-\mathbf{k})\, e^{\mathrm{i}\,\mathbf{k}\,\mathbf{x}} \right) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \\ &= \int_{\mathbb{R}^{d-1}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a(\mathbf{k})\, e^{\mathrm{i}\,\mathbf{k}\,\mathbf{x}} + a^\dagger(\mathbf{k})\, e^{-\mathrm{i}\,\mathbf{k}\,\mathbf{x}} \right) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \end{aligned} \tag{3.16}$$

where in the second line we have transformed the second term under the integral via $\mathbf{k} \mapsto -\mathbf{k}$. Using further that $[:H:, a(\mathbf{k})] = -\omega_{\mathbf{k}}\, a(\mathbf{k})$ and that $[:H:, a^\dagger(\mathbf{k})] = \omega_{\mathbf{k}}\, a^\dagger(\mathbf{k})$, which can be derived from (3.13) and (3.10), we can compute

$$\begin{aligned}\Phi(x) = e^{\mathrm{i}\,t\,[:H:,-]}\,\Phi(\mathbf{x}) &= \int_{\mathbb{R}^{d-1}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(e^{\mathrm{i}\,t\,[:H:,-]}\, a(\mathbf{k})\, e^{\mathrm{i}\,\mathbf{k}\,\mathbf{x}} + e^{\mathrm{i}\,t\,[:H:,-]}\, a^\dagger(\mathbf{k})\, e^{-\mathrm{i}\,\mathbf{k}\,\mathbf{x}} \right) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \\ &= \int_{\mathbb{R}^{d-1}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(e^{-\mathrm{i}\,\omega_{\mathbf{k}}\,t}\, a(\mathbf{k})\, e^{\mathrm{i}\,\mathbf{k}\,\mathbf{x}} + e^{\mathrm{i}\,\omega_{\mathbf{k}}\,t}\, a^\dagger(\mathbf{k})\, e^{-\mathrm{i}\,\mathbf{k}\,\mathbf{x}} \right) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad . \end{aligned} \tag{3.17}$$

Introducing the relativistic Fourier momentum

$$k := \begin{pmatrix} \omega_{\mathbf{k}} \\ \mathbf{k} \end{pmatrix} \in \mathbb{R}^d \tag{3.18}$$

that, as a consequence of (3.5), satisfies the relativistic energy-momentum relation (also called *on-shell condition* in the context of QFT)

$$k^2 = \eta_{\mu\nu}\, k^\mu\, k^\nu = -m^2 \quad , \tag{3.19}$$

we can write the Heisenberg picture field operator in a very compact form

$$\Phi(x) = \int_{\mathbb{R}^{d-1}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a(\mathbf{k})\, e^{\mathrm{i}\,k\,x} + a^\dagger(\mathbf{k})\, e^{-\mathrm{i}\,k\,x} \right) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \tag{3.20}$$

where the exponents involve the Minkowski inner product $k\,x := \eta_{\mu\nu}\,k^\mu\,x^\nu$. Last but not least, we note that this field operator satisfies the Klein-Gordon equation

$$\begin{aligned}(-\partial^2+m^2)\Phi(x) &= \int_{\mathbb{R}^{d-1}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left(a(\mathbf{k})\,(-\partial^2+m^2)e^{\mathrm{i}\,k\,x}+a^\dagger(\mathbf{k})\,(-\partial^2+m^2)e^{-\mathrm{i}\,k\,x}\right)\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}}\\ &= \int_{\mathbb{R}^{d-1}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left(a(\mathbf{k})\,(k^2+m^2)e^{\mathrm{i}\,k\,x}+a^\dagger(\mathbf{k})\,(k^2+m^2)e^{-\mathrm{i}\,k\,x}\right)\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}}\\ &= 0 \quad , \end{aligned} \tag{3.21}$$

where in the last step we have used the on-shell condition (3.19). Hence, the quantum dynamics of the free Klein-Gordon field is governed by the same equation that arises as the Euler-Lagrange equation for the classical free Klein-Gordon field. It is easy to check that the Heisenberg picture operator corresponding to the canonical momentum is given by the time derivative $\Pi(x) = \dot{\Phi}(x)$ of (3.20), hence it contains no new information.

We conclude this section by studying the commutation relation $[\Phi(x), \Phi(y)]$ between two Heisenberg picture field operators. In contrast to the equal-time commutation relations in (3.1), this now involves two arbitrary spacetime points $x, y \in \mathbb{R}^d$ that do not necessarily have the same time coordinates. With a short calculation using (3.20) and (3.10), one obtains

$$\boxed{\big[\Phi(x),\Phi(y)\big] = \int_{\mathbb{R}^{d-1}} \frac{1}{2\,\omega_{\mathbf{k}}}\left(e^{\mathrm{i}\,k\,(x-y)}-e^{-\mathrm{i}\,k\,(x-y)}\right)\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad .} \tag{3.22}$$

Observe that the right-hand side is just a complex number, which is sometimes called the *commutator function* and denoted by

$$\Delta(x-y) := \big[\Phi(x),\Phi(y)\big] \quad . \tag{3.23}$$

We can rewrite this as an integral over all $k = (k^0, \mathbf{k}) \in \mathbb{R}^d$ (not subject to the on-shell condition (3.19))

$$\Delta(x-y) = \int_{\mathbb{R}^d} \delta(k^2+m^2)\;\Theta(k^0)\;\left(e^{\mathrm{i}\,k\,(x-y)}-e^{-\mathrm{i}\,k\,(x-y)}\right)\frac{\mathrm{d}k}{(2\pi)^{d-1}} \quad , \tag{3.24}$$

where

$$\Theta(k^0) = \begin{cases} 1 & , \text{ for } k^0 > 0 \quad , \\ \frac{1}{2} & , \text{ for } k^0 = 0 \quad , \\ 0 & , \text{ for } k^0 < 0 \quad , \end{cases} \tag{3.25}$$

denotes the Heaviside step function. Indeed, using $k^2+m^2 = -(k^0)^2+\mathbf{k}^2+m^2 = -(k^0)^2+\omega_{\mathbf{k}}^2$ and standard properties of the Dirac delta function, one finds that

$$\begin{aligned}\delta(k^2+m^2)\,\Theta(k^0) &= \delta\Big((\omega_{\mathbf{k}}+k^0)\,(\omega_{\mathbf{k}}-k^0)\Big)\,\Theta(k^0)\\ &= \frac{1}{\omega_{\mathbf{k}}+k^0}\,\delta(\omega_{\mathbf{k}}-k^0)\,\Theta(k^0) \quad , \end{aligned} \tag{3.26}$$

from which one shows that (3.24) agrees with (3.22) by carrying out the integral over k^0. The advantage of the expression (3.24) is that it is manifestly invariant under proper and orthochronous Poincaré transformations, see Section 1.4.

With these preparations, we can show that the commutator function $\Delta(x-y) = [\Phi(x), \Phi(y)] = 0$ vanishes whenever the points $x, y \in \mathbb{R}^d$ are spacelike separated, i.e. $(x-y)^2 > 0$. It is a standard exercise in special relativity to show that for each pair of spacelike separated points there exists a proper and orthochronous Poincaré transformation such that the transformed difference $x' - y' = (0, \mathbf{z})$ has a zero time component. (In words, for each pair of spacelike separated points there exists a choice of coordinates such that the associated events happen at the same time.) Since the commutator function (3.24), and hence also (3.22), is invariant under such Poincaré transformations, we can transform to the new coordinates and find that

$$\Delta(x-y) = \Delta'(x'-y') = \int_{\mathbb{R}^{d-1}} \frac{1}{2\,\omega_{\mathbf{k}'}} \left(e^{\mathrm{i}\,\mathbf{k}'\,\mathbf{z}} - e^{-\mathrm{i}\,\mathbf{k}'\,\mathbf{z}} \right) \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^{d-1}} = 0 \quad , \quad \text{for } (x-y)^2 > 0 \quad , \tag{3.27}$$

because the integrand is an odd function in $\mathbf{k}'$ and hence it integrates to zero. This result can be interpreted as a statement about the *causality* of the quantum Klein-Gordon field. Indeed, the main feature of commuting operators in quantum theory is that they are independent in the sense that they do not influence or alter the measurement of each other, which in our context means that the field operators $\Phi(x)$ and $\Phi(y)$ associated with spacelike separated points are independent of each other.

3.2 Poincaré symmetry

Our canonical quantization procedure in Section 3.1 heavily relies on choosing a splitting of spacetime $x = (t, \mathbf{x})$ into time and space. These two concepts are then used very differently in the construction, hence it is a priori unclear if the free quantum Klein-Gordon field inherits the Poincaré symmetries of its classical analog from Example 2.4. Note that there are already some indications that the free quantum Klein-Gordon field is compatible with the laws of special relativity, for instance it satisfies the Poincaré invariant Klein-Gordon equation (3.21) and relativistic causality, but we still have to make more precise in which sense it admits Poincaré symmetries.

Recall that symmetries in quantum theory are realized by unitary operators U that act on other operators A via the adjoint action $A \mapsto UAU^\dagger$. Given a continuous family of symmetries $U_\beta = e^{\mathrm{i}\,\beta_a X^a}$ that is obtained by exponentiating Hermitian operators $X^1, \dots, X^N$ (called the *generators*) with parameters $\beta = (\beta_1, \dots, \beta_N) \in \mathbb{R}^N$, we can Taylor expand the adjoint action

$$U_\beta\, A\, U_\beta^\dagger = e^{\mathrm{i}\,\beta_a X^a}\, A\, e^{-\mathrm{i}\,\beta_a X^a} = A + \mathrm{i}\,\beta_a\,[X^a, A] + \mathcal{O}(\beta^2) \tag{3.28}$$

and find that the infinitesimal transformation of A is given by the commutator

$$\delta_\beta A = \mathrm{i}\,\beta_a [X^a, A] \quad . \tag{3.29}$$

When discussing symmetries in QFT, we shall focus mainly on such infinitesimal transformations. Many noninfinitesimal symmetries, but crucially not all, then follow by exponentiating infinitesimal transformations. For Poincaré transformations, one can show that every proper and orthochronous transformation (recall Section 1.4) can be obtained by exponentiating generators, but more general Poincaré transformations such as time-reversal $(t, \mathbf{x}) \mapsto (-t, \mathbf{x})$ do not arise in this way.

You might now ask: From where do we get symmetry generators? In classical field theory, Noether's Theorem 2.6 provides an answer: A continuous family of symmetries gives rise to a conserved Noether current J^μ and hence to a conserved charge $Q = \int_{\mathbb{R}^{d-1}} J^0(x)\,\mathrm{d}\mathbf{x}$. Passing to the Hamiltonian formalism for classical field theory, one can show that Q defines the corresponding (classical) symmetry generator that acts on the field and its canonical momentum via the Poisson bracket $\{Q, -\}$. This suggests that we should try to obtain symmetry generators in QFT by quantizing their associated classical Noether charges.

Let us illustrate how this works for the case of Poincaré transformations. The associated (classical) Noether charges for the free Klein-Gordon field have been computed in Examples 2.9 and 2.10. Let us start with the relativistic momentum P^μ from (2.37), which is the conserved charge associated with spacetime translations. When written in terms of the field Φ and its conjugate momentum Π, the components read as

$$P^0 = \int_{\mathbb{R}^{d-1}} \frac{1}{2}\Big(\Pi^2 + (\nabla\Phi)^2 + m^2\,\Phi^2\Big)\,\mathrm{d}\mathbf{x} \quad , \qquad P^i = -\int_{\mathbb{R}^{d-1}} \Pi\,\partial^i\Phi\,\mathrm{d}\mathbf{x} \quad . \tag{3.30}$$

Note that $P^0 = H$ agrees with the Hamiltonian, whose normal ordered quantization was worked out in Section 3.1 and is given by

$$:P^0: = :H: = \int_{\mathbb{R}^{d-1}} \omega_{\mathbf{k}}\; a^\dagger(\mathbf{k})\,a(\mathbf{k})\;\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad . \tag{3.31}$$

Using the same Fourier transformation techniques as in Section 3.1, we can also determine a quantization of the spatial components P^i. Let me illustrate briefly the relevant steps. Inserting the Fourier expansions (3.3) and (3.8) into the formula for P^i yields

$$\begin{aligned}
P^i &= \int_{\mathbb{R}^{d-1}} \widetilde{\Pi}(\mathbf{k})\,\mathrm{i}\,k^i\,\widetilde{\Phi}(-\mathbf{k})\,\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \\
&= \int_{\mathbb{R}^{d-1}} \frac{k^i}{2}\,\Big(a(\mathbf{k}) - a^\dagger(-\mathbf{k})\Big)\,\Big(a(-\mathbf{k}) + a^\dagger(\mathbf{k})\Big)\,\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \\
&= \int_{\mathbb{R}^{d-1}} \frac{k^i}{2}\,\Big(a(\mathbf{k})\,a(-\mathbf{k}) - a^\dagger(-\mathbf{k})\,a(-\mathbf{k}) + a(\mathbf{k})\,a^\dagger(\mathbf{k}) - a^\dagger(-\mathbf{k})\,a^\dagger(\mathbf{k})\Big)\,\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \\
&= \int_{\mathbb{R}^{d-1}} \frac{k^i}{2}\,\Big(a^\dagger(\mathbf{k})\,a(\mathbf{k}) + a(\mathbf{k})\,a^\dagger(\mathbf{k})\Big)\,\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad ,
\end{aligned} \tag{3.32}$$

where in the last step we have used that $k^i\,a(\mathbf{k})\,a(-\mathbf{k})$ and $k^i\,a^\dagger(-\mathbf{k})\,a^\dagger(\mathbf{k})$ are odd functions in $\mathbf{k}$, hence the corresponding integrals vanish. Applying normal ordering (see Definition 3.1), we obtain

$$:P^i: = \int_{\mathbb{R}^{d-1}} k^i\; a^\dagger(\mathbf{k})\,a(\mathbf{k})\,\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \tag{3.33}$$

hence the normal ordered quantization of the relativistic momentum is simply given by

$$:P^\mu: = \int_{\mathbb{R}^{d-1}} k^\mu\; a^\dagger(\mathbf{k})\,a(\mathbf{k})\,\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \tag{3.34}$$

with the relativistic Fourier momentum $k = (\omega_{\mathbf{k}}, \mathbf{k})$ introduced in (3.18). With a similar but more involved calculation, one obtains the normal ordered quantization of the relativistic angular momentum $L^{\rho\nu}$ from (2.44). Its components read explicitly as

$$:L^{i0}: = -:L^{0i}: = \int_{\mathbb{R}^{d-1}} a^\dagger(\mathbf{k}) \left(\mathrm{i}\,\omega_{\mathbf{k}} \frac{\partial}{\partial k_i} + \frac{\mathrm{i}}{2} \frac{\partial \omega_{\mathbf{k}}}{\partial k_i} \right) a(\mathbf{k}) \, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \tag{3.35a}$$

and

$$:L^{ij}: = \int_{\mathbb{R}^{d-1}} a^\dagger(\mathbf{k}) \left(\mathrm{i}\,k^j \frac{\partial}{\partial k_i} - \mathrm{i}\,k^i \frac{\partial}{\partial k_j} \right) a(\mathbf{k}) \, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad . \tag{3.35b}$$

Remark 3.3. One can check that these operators satisfy the *Poincaré Lie algebra* relations

$$\begin{aligned} \big[:P^\mu:, :P^\nu:\big] &= 0 \quad , \qquad \big[:L^{\mu\nu}:, :P^\rho:\big] = \mathrm{i}\,\big(\eta^{\mu\rho}\,:P^\nu: - \eta^{\nu\rho}\,:P^\mu:\big) \quad , \\ \big[:L^{\mu\nu}:, :L^{\rho\sigma}:\big] &= \mathrm{i}\,\big(\eta^{\nu\rho}\,:L^{\mu\sigma}: - \eta^{\mu\rho}\,:L^{\nu\sigma}: - \eta^{\nu\sigma}\,:L^{\mu\rho}: + \eta^{\mu\sigma}\,:L^{\nu\rho}:\big) \quad , \end{aligned} \tag{3.36}$$

that characterize infinitesimal Poincaré transformations. We shall introduce Lie algebras more formally later in this module, so for the moment this is just a side-remark. More details about the Lorentz and Poincaré Lie algebras can be found e.g. in the textbook by Maggiore (Chapter 2).

With these preparations, we can now investigate how the Heisenberg picture field operator (3.20) transforms under (infinitesimal) Poincaré transformations and thereby answer affirmatively the question whether the quantum Klein-Gordon field inherits the Poincaré symmetries from its classical counterpart. For the transformations induced by the generators $:P^\mu:$, we find

$$\begin{aligned} \delta_b \Phi(x) &= \mathrm{i}\,b_\mu \big[:P^\mu:, \Phi(x)\big] \\ &= \mathrm{i}\,b_\mu \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{k^\mu}{\sqrt{2\,\omega_{\mathbf{q}}}} \Big[a^\dagger(\mathbf{k})\,a(\mathbf{k}), a(\mathbf{q})\,e^{\mathrm{i}\,q\,x} + a^\dagger(\mathbf{q})\,e^{-\mathrm{i}\,q\,x} \Big] \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \frac{\mathrm{d}\mathbf{q}}{(2\pi)^{d-1}} \\ &= \mathrm{i}\,b_\mu \int_{\mathbb{R}^{d-1}} \frac{k^\mu}{\sqrt{2\,\omega_{\mathbf{k}}}} \Big(- a(\mathbf{k})\,e^{\mathrm{i}\,k\,x} + a^\dagger(\mathbf{k})\,e^{-\mathrm{i}\,k\,x} \Big) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \\ &= -b_\mu\,\partial^\mu \Phi(x) \quad , \end{aligned} \tag{3.37}$$

where in the third equality we used the commutation relations (3.10) for the annihilation and creation operators. We recognize this as the infinitesimal spacetime translations from Example 2.9. Hence, the operators $:P^\mu:$ are the generators for translations. With a more involved calculation, one finds for the transformation induced by the generators $:L^{\nu\rho}:$ that

$$\delta_\omega \Phi(x) = \frac{\mathrm{i}}{2}\,\omega_{\rho\nu} \big[:L^{\nu\rho}:, \Phi(x)\big] = -\omega^\mu{}_\nu\,x^\nu\,\partial_\mu \Phi(x) \quad , \tag{3.38}$$

which we recognize as the infinitesimal Lorentz transformations from Example 2.10. Hence, the operators $:L^{\nu\rho}:$ are the generators for Lorentz transformations.

Summing up, we have found operators $:P^\mu:$ and $:L^{\nu\rho}:$ that play the role of generators for infinitesimal Poincaré transformations. These generators implement Poincaré symmetries at the level of the quantum field $\Phi(x)$ in a way that is characteristic for scalar fields.

3.3 Hilbert space and particle interpretation

To obtain a representation of the Heisenberg picture field operators (3.20) on a Hilbert space $\mathcal{H}$, we can proceed in analogy to the quantum harmonic oscillator and build a Hilbert space using the annihilation and creation operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$. Such type of Hilbert space is often called *Fock space* in the literature.

We start by introducing a ground state $|0\rangle \in \mathcal{H}$, called the *vacuum state* in the context of QFT, which is characterized by the property that it is annihilated by all annihilation operators, i.e.

$$a(\mathbf{k})|0\rangle = 0 \quad , \tag{3.39}$$

for all $\mathbf{k} \in \mathbb{R}^{d-1}$, and the usual normalization condition $\langle 0|0\rangle = 1$ for states. Note that the vacuum state is an eigenstate of the normal ordered relativistic momentum (3.34) and relativistic angular momentum (3.35) operators with zero eigenvalues,

$$:P^\mu:|0\rangle = 0 \quad , \qquad :L^{\nu\rho}:|0\rangle = 0 \quad . \tag{3.40}$$

In words, the vacuum has zero relativistic momentum and angular momentum, which is a feature resulting from our normal ordering prescription.

Further states are then obtained by acting with the creation operators $a^\dagger(\mathbf{k})$ on the vacuum state $|0\rangle$. Acting with n creation operators results in the state

$$|k_1, \dots, k_n\rangle := \sqrt{2\,\omega_{\mathbf{k}_1}} \cdots \sqrt{2\,\omega_{\mathbf{k}_n}}\; a^\dagger(\mathbf{k}_1) \cdots a^\dagger(\mathbf{k}_n)|0\rangle \;\in\; \mathcal{H} \quad , \tag{3.41}$$

which we label by a family of relativistic Fourier momenta $k_a = (\omega_{\mathbf{k}_a}, \mathbf{k}_a) \in \mathbb{R}^d$, for $a = 1, \dots, n$, that satisfy the on-shell condition $k_a^2 = -m^2$ from (3.19). The factors $\sqrt{2\,\omega_{\mathbf{k}_a}}$ are a convenient choice of normalization. (As we explain below, they are useful to make manifest the Poincaré symmetry of our QFT.) The physical interpretation of (3.41) is that of an *n-particle state*. From the fact that any two creation operators commute, i.e. $a^\dagger(\mathbf{q})\, a^\dagger(\mathbf{k}) = a^\dagger(\mathbf{k})\, a^\dagger(\mathbf{q})$, we find that $|k_1, \dots, k_n\rangle$ is symmetric under the exchange of any pair of k's, hence the particles are bosons.

Can one say more about these particles? For instance, what are their masses and spins? To answer these questions, it suffices to analyze a single particle state $|k\rangle = \sqrt{2\,\omega_{\mathbf{k}}}\, a^\dagger(\mathbf{k})|0\rangle$. One easily checks that this is an eigenstate of the normal ordered relativistic momentum operator (3.34) with eigenvalue

$$:P^\mu:|k\rangle = k^\mu\, |k\rangle \tag{3.42}$$

returning the relativistic Fourier momentum k. Hence, one interprets $|k\rangle$ as a state that describes a single quantum particle with fixed relativistic momentum $k = (\omega_{\mathbf{k}}, \mathbf{k})$. (Note that the energy of the particle is positive.) The on-shell condition $k^2 = -m^2$ then tells us that the parameter $m \geq 0$ from the Klein-Gordon action defines the mass of the particle. To say something about the spin of the particle, we act with the relativistic angular momentum operator (3.35) on $|k\rangle$ and observe that the spatial components $:L^{ij}:|k\rangle$ vanish when we take (informally) the limit $\mathbf{k} \to \mathbf{0}$. Note that this limit models the scenario in which the particle is not moving, hence it detects the intrinsic

angular momentum, a.k.a. the spin. Since the latter vanishes, the particles associated with the free Klein-Gordon quantum field have spin 0.

We shall now investigate the inner product on the Hilbert space. Since the states (3.41) are defined by creation operators, the inner product can be determined from the commutation relations (3.10) and the definition of the vacuum state (3.39). In full generality, one finds that

$$\langle q_1, \dots, q_m | k_1, \dots, k_n \rangle = \begin{cases} \sum_{\sigma \in S_n} \prod_{a=1}^{n} \Big((2\pi)^{d-1} \, 2\, \omega_{\mathbf{k}_a} \, \delta(\mathbf{k}_a - \mathbf{q}_{\sigma(a)}) \Big) & , \text{ for } n = m \quad , \\ 0 & , \text{ for } n \neq m \quad , \end{cases} \tag{3.43}$$

where S_n denotes the permutation group on n letters. In particular, any two states with a different number of particles are orthogonal to each other. In case you find this general formula hard to read, it helps to look at some simple examples

$$\begin{aligned} \langle q | k \rangle &= (2\pi)^{d-1} \, 2\, \omega_{\mathbf{k}} \, \delta(\mathbf{k} - \mathbf{q}) \quad , \\ \langle q_1, q_2 | k_1, k_2 \rangle &= (2\pi)^{2(d-1)} \, 2\, \omega_{\mathbf{k}_1} \, 2\, \omega_{\mathbf{k}_2} \Big(\delta(\mathbf{k}_1 - \mathbf{q}_1) \, \delta(\mathbf{k}_2 - \mathbf{q}_2) + \delta(\mathbf{k}_1 - \mathbf{q}_2) \, \delta(\mathbf{k}_2 - \mathbf{q}_1) \Big) \quad . \end{aligned} \tag{3.44}$$

Working with the inner products in (3.43) requires some care because of the appearance of Dirac delta functions, which can cause divergences such as e.g. in $\langle k | k \rangle = (2\pi)^{d-1} \, 2\, \omega_{\mathbf{0}} \, \delta(\mathbf{0})$. The way how one can make these inner products mathematically precise is by considering so-called *wave packet states* instead of the states $|k\rangle$ that, as we have seen, are too singular. Let me illustrate the key idea for 1-particle states. (The n-particle wave packet states are defined similarly.) Consider the following state

$$|\psi\rangle := \int_{\mathbb{R}^{d-1}} \widetilde{\psi}(\mathbf{k}) \, |k\rangle \, \frac{\mathrm{d}\mathbf{k}}{2\, \omega_{\mathbf{k}} \, (2\pi)^{d-1}} \tag{3.45}$$

that is given by integrating $|k\rangle$ over a wave packet $\widetilde{\psi}(\mathbf{k})$ in the Fourier space. (Think, for instance, of a Gaussian that is peaked around your favorite momentum $\mathbf{k}$.) Physically, the state $|\psi\rangle$ describes a quantum particle that does not have a sharp relativistic momentum, but rather is in a superposition of different momenta. Note that the integration measure has been chosen such that it is invariant under proper and orthochronous Poincaré transformations, which follows from the same argument used in (3.26). The inner product between two 1-particle wave packet states is then given by

$$\begin{aligned} \langle \chi | \psi \rangle &= \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \widetilde{\chi}(\mathbf{q})^* \, \widetilde{\psi}(\mathbf{k}) \, \langle \mathbf{q} | \mathbf{k} \rangle \, \frac{1}{2\, \omega_{\mathbf{q}} \, 2\, \omega_{\mathbf{k}}} \, \frac{\mathrm{d}\mathbf{q}}{(2\pi)^{d-1}} \, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \\ &= \int_{\mathbb{R}^{d-1}} \widetilde{\chi}(\mathbf{k})^* \, \widetilde{\psi}(\mathbf{k}) \, \frac{1}{2\, \omega_{\mathbf{k}}} \, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \end{aligned} \tag{3.46}$$

where we recognize again the Poincaré invariant measure. Note that this expression is well-defined, provided that the two wave packets are square-integrable with respect to the measure. We can now finally justify our choice of normalization in (3.41): The factors of $\sqrt{2\, \omega_{\mathbf{k}_a}}$ are needed in order to make the wave packet inner product (3.46) manifestly Poincaré invariant. To conclude, let us note

that the analog of the inner product (3.43) for multiparticle wave packet states reads as

$$\langle \chi_1, \dots, \chi_m | \psi_1, \dots, \psi_n \rangle = \begin{cases} \sum_{\sigma \in S_n} \prod_{a=1}^{n} \langle \chi_{\sigma(a)} | \psi_a \rangle & , \text{ for } n = m \quad , \\ 0 & , \text{ for } n \neq m \quad . \end{cases} \tag{3.47}$$

Again, this expression is well-defined, provided that all wave packets are square-integrable with respect to the measure.

3.4 Correlation functions and Feynman propagator

Using the Hilbert space representation from the previous section, we can now determine and study the correlation functions of the Heisenberg picture field operators $\Phi(x)$. One defines the *n-point correlation function*, which is sometimes also called the *n-point Wightman function*, to be the vacuum expectation value

$$W_n(x_1, \dots, x_n) := \langle 0 | \Phi(x_1) \cdots \Phi(x_n) | 0 \rangle \tag{3.48}$$

of the product of $n \geq 1$ quantum fields at different points $x_1, \dots, x_n \in \mathbb{R}^d$ of Minkowski spacetime. Using the presentation of $\Phi(x)$ in terms of creation and annihilation operators from (3.20), the definition of the vacuum state (3.39) and the commutation relations (3.10), one can compute these correlation functions explicitly. For low n, one finds

$$W_1(x) = \langle 0 | \Phi(x) | 0 \rangle = 0 \quad , \tag{3.49a}$$

$$W_2(x, y) = \langle 0 | \Phi(x) \Phi(y) | 0 \rangle = \int_{\mathbb{R}^{d-1}} \frac{1}{2\,\omega_{\mathbf{k}}} \, e^{\mathrm{i}\, k\,(x-y)} \, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \tag{3.49b}$$

where we recognize again the Poincaré invariant measure. In particular, this implies that the 1- and 2-point functions are invariant under proper and orthochronous Poincaré transformations, i.e. $W_1'(x') = W_1(x)$ and $W_2'(x', y') = W_2(x, y)$. (The same will hold true for the higher n-point functions.) As an exercise, you can try to compute the 3- and 4-point correlation functions. In general, it turns out that the n-point function $W_n(x_1, \dots, x_n)$ of a free QFT is determined completely by the 2-point function $W_2(x, y)$, but we are not going to prove this now. Using again the trick in (3.26), we can also rewrite the 2-point function as an integral over $k \in \mathbb{R}^d$

$$W_2(x, y) = \int_{\mathbb{R}^d} \delta(k^2 + m^2) \; \Theta(k^0) \, e^{\mathrm{i}\, k\,(x-y)} \, \frac{\mathrm{d}k}{(2\pi)^{d-1}} \quad , \tag{3.50}$$

from which Poincaré invariance is even better visible. Note that $W_2(x, y)$ satisfies the Klein-Gordon equation in both entries, i.e.

$$(-\partial_x^2 + m^2) W_2(x, y) = 0 = (-\partial_y^2 + m^2) W_2(x, y) \quad , \tag{3.51}$$

where the subscript indicates the coordinates along which one takes partial derivatives, and that the commutator function (3.22) is the following difference

$$\Delta(x - y) = W_2(x, y) - W_2(y, x) \quad , \tag{3.52}$$

which is of course clear from the definition $W_2(x,y) = \langle 0|\Phi(x)\Phi(y)|0\rangle$ and the normalization condition $\langle 0|0\rangle$ of the vacuum state. In contrast to the causality of the commutator function (see (3.27)), the 2-point function $W_2(x,y)$ does *not* vanish for spacelike separated point $x, y \in \mathbb{R}^d$. This means that the vacuum state $|0\rangle \in \mathcal{H}$ exhibits nonlocal features, which one may interpret as a quantum entanglement over spacelike separated distances. It is a common feature of QFT that the field operators behave local and causal, but the vacuum state is a nonlocal concept.

As we shall see in the next chapter, it turns out that in QFT a different type of n-point functions is more important, namely the so-called *time-ordered n-point functions*

$$G_n(x_1,\dots,x_n) := \langle 0|\mathsf{T}(\Phi(x_1)\cdots\Phi(x_n))|0\rangle \quad . \tag{3.53}$$

The symbol $\mathsf{T}(\cdots)$ denotes the time-ordered product of field operators, which means that field operators that are located at later times are placed to the left of field operators that are located at earlier times. For example, the time-ordered product of two field operators is given by

$$\begin{aligned}\mathsf{T}(\Phi(x)\Phi(y)) &= \begin{cases}\Phi(x)\,\Phi(y) & , \text{ if } x^0 \geq y^0 \quad , \\ \Phi(y)\,\Phi(x) & , \text{ if } y^0 \geq x^0 \quad , \end{cases} \\ &= \Theta(x^0-y^0)\,\Phi(x)\,\Phi(y) + \Theta(y^0-x^0)\,\Phi(y)\,\Phi(x) \quad . \end{aligned} \tag{3.54}$$

Note that the two cases agree for $x^0 = y^0$, which is a consequence of the equal-time commutation relations (3.1). A general formula for $\mathsf{T}(\Phi(x_1)\cdots\Phi(x_n))$ can be obtained by noting that for any tuple $x_1,\dots,x_n \in \mathbb{R}^d$ of n-points in Minkowski spacetime there exists a permutation $\sigma \in S_n$, such that the time coordinates $t_a = x^0_a$ are ordered according to $x^0_{\sigma(1)} \geq x^0_{\sigma(2)} \geq \cdots \geq x^0_{\sigma(n)}$. Then the time-ordered product is given by

$$\mathsf{T}(\Phi(x_1)\cdots\Phi(x_n)) = \Phi(x_{\sigma(1)})\cdots\Phi(x_{\sigma(n)}) \quad . \tag{3.55}$$

Remark 3.4. It is not immediately clear that the time-ordered product is invariant under proper and orthochronous Poincaré transformations because its definition uses a fixed choice of time coordinate $x = (t, \mathbf{x})$. A nice property of such Poincaré transformations, which is relatively easy to prove but not shown here, is that they can invert the time-ordering of two points $x, y \in \mathbb{R}^d$, i.e. $x^0 > y^0$ transforms to $x'^0 < y'^0$, only if these points are spacelike separated, i.e. $(x-y)^2 > 0$. Due to causality (3.27), we know that any two field operators located at spacelike separated points commute with each other, hence the time-ordered products computed in the two different coordinate systems x and x' coincide. This implies that T is indeed invariant under proper and orthochronous Poincaré transformations.

Let us compute the first two examples of the time-ordered n-point functions. For the time-ordered 1-point function, we find again zero

$$\langle 0|\mathsf{T}(\Phi(x))|0\rangle = \langle 0|\Phi(x)|0\rangle = W_1(x) = 0 \quad . \tag{3.56a}$$

The time-ordered 2-point function, which is called the *Feynman propagator* Δ_F, is more interesting

and, using (3.54), it is given by

$$\Delta_F(x-y) := \langle 0|\mathsf{T}\big(\Phi(x)\,\Phi(y)\big)|0\rangle = \begin{cases} W_2(x,y) & \text{, if } x^0 \geq y^0 \\ W_2(y,x) & \text{, if } y^0 \geq x^0 \end{cases},$$

$$= \Theta(x^0-y^0)\,W_2(x,y) + \Theta(y^0-x^0)\,W_2(y,x) \quad , \tag{3.56b}$$

where W_2 is the 2-point correlation function. Note that in both cases the two arguments of W_2 are ordered such that the later spacetime point is to the left of the earlier one, which is of course precisely what time-ordering does. We will see later that all other time-ordered n-point functions can be determined from $\Delta_F(x-y)$ via Wick's theorem, see Theorem 4.1 and Corollary 4.3. The formula above for the time-ordered 2-point function is not very useful for calculations because it involves case distinctions. Using the concept of *contour integration*, there is a clever way to rewrite the Feynman propagator $\Delta_F(x-y)$ in terms of an integral formula over the relativistic Fourier momenta $k \in \mathbb{R}^d$. Let me first state the result and then show you how to prove it.

Proposition 3.5. *The time-ordered 2-point function/Feynman propagator can be written as the limit $\epsilon \to 0$ of the Fourier integral*

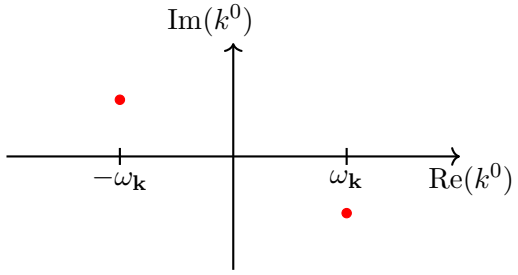
$$\Delta_F(x-y) = \langle 0|\mathsf{T}\big(\Phi(x)\,\Phi(y)\big)|0\rangle = \lim_{\epsilon\to 0} \int_{\mathbb{R}^d} \frac{-\mathrm{i}}{k^2+m^2-\mathrm{i}\,\epsilon}\, e^{\mathrm{i}\,k\,(x-y)}\, \frac{\mathrm{d}k}{(2\pi)^d} \quad , \tag{3.57}$$

where $\epsilon > 0$ is a positive parameter whose role will become clear in the proof below.

Proof. Throughout the proof, we assume that $\epsilon > 0$ is positive and sufficiently small so that terms of order ϵ^2 can be neglected. To compare the integral formula (3.57) with (3.56b), we carry out the integral over the 0-component k^0 of the relativistic Fourier momentum. Splitting into time and space components, the denominator reads as

$$k^2+m^2-\mathrm{i}\,\epsilon = -(k^0)^2+\omega_{\mathbf{k}}^2-\mathrm{i}\,\epsilon = -\big(k^0-\omega_{\mathbf{k}}+\mathrm{i}\,\epsilon'\big)\,\big(k^0+\omega_{\mathbf{k}}-\mathrm{i}\,\epsilon'\big) \quad , \tag{3.58}$$

where $\epsilon' = \epsilon/(2\omega_{\mathbf{k}}) > 0$ and we recall that terms of order ϵ^2 are neglected. This means that the integrand in (3.57) has two first order poles in k^0 that are located as follows in the complex plane:

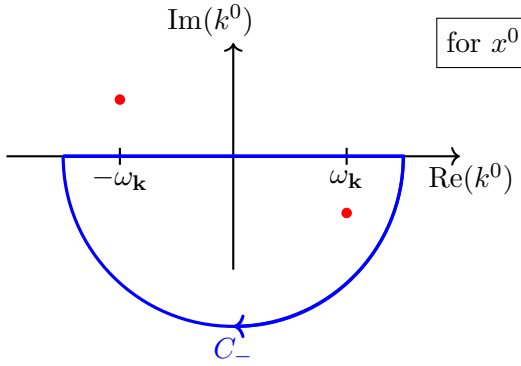


(3.59)

The idea is now to perform a contour integration in order to compute the k^0 integral. To figure out which contour to choose, note that the integrand in (3.57) also contains a complex exponential $e^{\mathrm{i}\,k\,(x-y)} = e^{-\mathrm{i}\,k^0\,(x^0-y^0)+\mathrm{i}\,\mathbf{k}\,(\mathbf{x}-\mathbf{y})}$. Since $k^0 = \mathrm{Re}(k^0)+\mathrm{i}\,\mathrm{Im}(k^0)$ was extended to a complex number, this yields a factor of the form

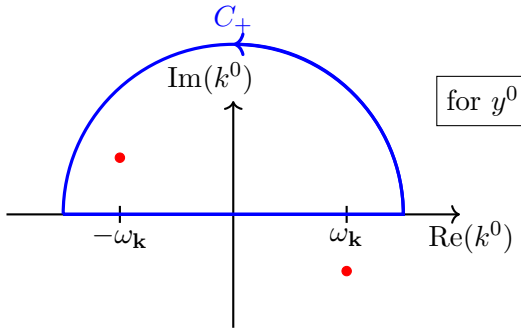
$$e^{-\mathrm{i}\,k^0\,(x^0-y^0)} = e^{-\mathrm{i}\,\mathrm{Re}(k^0)\,(x^0-y^0)+\mathrm{Im}(k^0)\,(x^0-y^0)} \quad . \tag{3.60}$$

This means that the integrand falls off in the imaginary direction $\mathrm{Im}(k^0)$ for the following cases: 1.) If $x^0 \geq y^0$, it falls off for $\mathrm{Im}(k^0) \to -\infty$. 2.) If $y^0 \geq x^0$, it falls off for $\mathrm{Im}(k^0) \to +\infty$. This justifies choosing (the infinite radius limit of) the following contours



(3.61a)

and



(3.61b)

Using Cauchy's residue theorem, one then finds for the case $x^0 \geq y^0$ that

$$\begin{aligned}\int_{\mathbb{R}} \frac{-\mathrm{i}}{k^2+m^2-\mathrm{i}\,\epsilon}\, e^{-\mathrm{i}\,k^0\,(x^0-y^0)}\,\frac{\mathrm{d}k^0}{2\pi} &= \oint_{C_-} \frac{\mathrm{i}}{(k^0-\omega_{\mathbf{k}}+\mathrm{i}\,\epsilon')\,(k^0+\omega_{\mathbf{k}}-\mathrm{i}\,\epsilon')}\, e^{-\mathrm{i}\,k^0\,(x^0-y^0)}\,\frac{\mathrm{d}k^0}{2\pi} \\ &= \frac{1}{2(\omega_{\mathbf{k}}-\mathrm{i}\,\epsilon')}\, e^{-\mathrm{i}\,(\omega_{\mathbf{k}}-\mathrm{i}\,\epsilon')\,(x^0-y^0)} \\ &\xrightarrow{\epsilon\to 0} \frac{1}{2\omega_{\mathbf{k}}}\, e^{-\mathrm{i}\,\omega_{\mathbf{k}}\,(x^0-y^0)}\end{aligned} \tag{3.62}$$

and similarly for the case $y^0 \geq x^0$ that

$$\begin{aligned}\int_{\mathbb{R}} \frac{-\mathrm{i}}{k^2+m^2-\mathrm{i}\,\epsilon}\, e^{-\mathrm{i}\,k^0\,(x^0-y^0)}\,\frac{\mathrm{d}k^0}{2\pi} &= \oint_{C_+} \frac{\mathrm{i}}{(k^0-\omega_{\mathbf{k}}+\mathrm{i}\,\epsilon')\,(k^0+\omega_{\mathbf{k}}-\mathrm{i}\,\epsilon')}\, e^{-\mathrm{i}\,k^0\,(x^0-y^0)}\,\frac{\mathrm{d}k^0}{2\pi} \\ &= \frac{1}{2(\omega_{\mathbf{k}}-\mathrm{i}\,\epsilon')}\, e^{+\mathrm{i}\,(\omega_{\mathbf{k}}-\mathrm{i}\,\epsilon')\,(x^0-y^0)} \\ &\xrightarrow{\epsilon\to 0} \frac{1}{2\omega_{\mathbf{k}}}\, e^{+\mathrm{i}\,\omega_{\mathbf{k}}\,(x^0-y^0)} = \frac{1}{2\omega_{\mathbf{k}}}\, e^{-\mathrm{i}\,\omega_{\mathbf{k}}\,(y^0-x^0)}\end{aligned} \quad . \tag{3.63}$$

Inserting these back into (3.57), and recalling the formula (3.49b) for the 2-point correlation function W_2, then shows that (3.57) agrees with (3.56b). □

To conclude this section, let us note that, in contrast to the 2-point correlation function (3.51), the Feynman propagator (3.57) is not a solution of the Klein-Gordon equation but rather a *Green's function*, i.e.

$$(-\partial_x^2 + m^2)\Delta_F(x-y) = -\mathrm{i}\,\delta(x-y) \quad . \tag{3.64}$$

What distinguishes the Feynman propagator from other choices of Green's functions, e.g. the retarded/advanced one that is obtained by moving instead of (3.59) both poles to the lower/upper complex half-plane, is the following feature: As visualized by the contours (3.61) in the proof above, the Feynman propagator $\Delta_F(x-y)$ propagates the positive frequencies $\omega_{\mathbf{k}}$ forward in time $x^0 \geq y^0$ and the negative frequencies $-\omega_{\mathbf{k}}$ backward in time $y^0 \geq x^0$.

Further reading

For more details about the quantization of the free Klein-Gordon field, see e.g. Nastase (Chapters 3 and 4), Srednicki (Chapter 3), Greiner/Reinhardt (Chapter 2) and Maggiore (Chapter 4.1) from our reading list in Section 1.3.

Chapter 4

Interacting quantum Klein-Gordon field

Interacting QFTs are much richer than the free QFTs from Chapter 3, and consequently also harder to describe. This chapter discusses the perturbative quantization of the interacting real Klein-Gordon field from Example 2.1 and uses these techniques to compute scattering amplitudes for quantum particle interactions.

4.1 Perturbation techniques

What characterizes an interacting QFT is that its underlying classical action functional contains terms that are of polynomial degree ≥ 3, which implies that the Euler-Lagrange equations are nonlinear partial differential equations. For instance, for a real scalar field Φ, one can consider as in Example 2.1 a potential term, say $V(\Phi) = \sum_{n\geq 3} \frac{\lambda_n}{n!}\, \Phi^n$ with some parameters $\lambda_n \in \mathbb{R}$ called the *coupling constants*, and write down the action

$$S[\Phi] = S_0[\Phi] + S_{\text{int}}[\Phi] = \int_{\mathbb{R}^d} -\frac{1}{2}\left(\partial^\mu \Phi\, \partial_\mu \Phi + m^2\, \Phi^2\right) \mathrm{d}x - \sum_{n\geq 3} \frac{\lambda_n}{n!} \int_{\mathbb{R}^d} \Phi^n \,\mathrm{d}x \quad . \tag{4.1}$$

The corresponding Hamiltonian reads as

$$\begin{aligned} H[\Pi, \Phi] &= H_0[\Pi, \Phi] + H_{\text{int}}[\Pi, \Phi] \\ &= \int_{\mathbb{R}^{d-1}} \frac{1}{2}\Big(\Pi^2 + (\nabla\Phi)^2 + m^2\, \Phi^2\Big)\, \mathrm{d}\mathbf{x} + \sum_{n\geq 3} \frac{\lambda_n}{n!} \int_{\mathbb{R}^{d-1}} \Phi^n \,\mathrm{d}\mathbf{x} \quad . \end{aligned} \tag{4.2}$$

In both expressions we use a subscript $_0$ to denote the quadratic parts, i.e. the action functional and Hamiltonian for the free Klein-Gordon field.

The same canonical quantization procedure as in Section 3.1 can be applied to this system, up to the point where we want to solve Heisenberg's equations. The issue there is that the higher-order powers Φ^n in the interaction Hamiltonian H_{int} make the commutators $[H_{\text{int}}, \Pi] \simeq \Phi^{n-1}$ nonlinear expressions in the Φ's, hence Heisenberg's equations are nonlinear differential equations whenever an interaction term is present. Making mathematical sense of such nonlinear differential equations for operators, which turn out to suffer from divergences that can not be cured by using only a

simple normal ordering prescription as in Definition 3.1, and determining solutions is an extremely difficult problem that has so far only been successfully addressed in very special cases.

When a nonlinear equation is too complicated, it is always a natural first step to work perturbatively by expanding order-by-order in the nonlinearities, with the hope to capture at least some features of the nonlinear equation. This is indeed the approach taken in most of the QFT literature and consequently also in our module. Ignoring for the moment all potential divergences that can arise in this expansion (these are treated later in Chapter 8 with renormalization techniques), what one does in the context of QFT is to write down a formal solution to Heisenberg's equation for the field operator

$$\Phi(x) = e^{\mathrm{i}\,t\,[H_0+H_{\mathrm{int}},-]}\,\Phi(\mathbf{x}) = e^{\mathrm{i}\,t\,(H_0+H_{\mathrm{int}})}\,\Phi(\mathbf{x})\,e^{-\mathrm{i}\,t\,(H_0+H_{\mathrm{int}})} \tag{4.3}$$

and expand this expression order-by-order in the coupling constants λ_n, hence in the interaction Hamiltonian H_{int}, that control the nonlinearities.

We can manipulate the expression (4.3) a bit further in order to bring it to a simpler and more useful form. The main idea is to factorize off the time evolution induced by the free Hamiltonian H_0, which we have explicitly solved in Section 3.1. This motivates us to rewrite (4.3) as follows

$$\begin{aligned}\Phi(x) &= e^{\mathrm{i}\,t\,(H_0+H_{\mathrm{int}})}\,e^{-\mathrm{i}\,t\,H_0}\,e^{\mathrm{i}\,t\,H_0}\,\Phi(\mathbf{x})\,e^{-\mathrm{i}\,t\,H_0}\,e^{\mathrm{i}\,t\,H_0}\,e^{-\mathrm{i}\,t\,(H_0+H_{\mathrm{int}})} \\ &= e^{\mathrm{i}\,t\,(H_0+H_{\mathrm{int}})}\,e^{-\mathrm{i}\,t\,H_0}\,\Phi_0(x)\,e^{\mathrm{i}\,t\,H_0}\,e^{-\mathrm{i}\,t\,(H_0+H_{\mathrm{int}})} \quad ,\end{aligned} \tag{4.4a}$$

where $\Phi_0(x)$ is the *free* Heisenberg picture field operator from (3.20), which I repeat here for your convenience

$$\Phi_0(x) = \int_{\mathbb{R}^{d-1}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a(\mathbf{k})\,e^{\mathrm{i}\,k\,x} + a^\dagger(\mathbf{k})\,e^{-\mathrm{i}\,k\,x} \right) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad . \tag{4.4b}$$

Note that normal ordering of H_0 is not required here since $[H_0,-] = [:H_0:,-]$ give the same right-hand sides of the free Heisenberg equation. You might now be tempted to identify the operator $e^{\mathrm{i}\,t\,H_0}\,e^{-\mathrm{i}\,t\,(H_0+H_{\mathrm{int}})}$ with the simpler one $e^{-\mathrm{i}\,t\,H_{\mathrm{int}}}$, but unfortunately this does not work because $[H_0, H_{\mathrm{int}}] \neq 0$ are noncommuting operators, hence the product of exponential functions is given by the Baker-Campbell-Hausdorff formula and not simply by summing the exponents.

There is however a clever trick to express the operator $e^{\mathrm{i}\,t\,H_0}\,e^{-\mathrm{i}\,t\,(H_0+H_{\mathrm{int}})}$ in a more useful way, which is called *Dyson's formula* and derived as follows: Taking the time derivative of this operator, one finds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\mathrm{i}\,t\,H_0}\,e^{-\mathrm{i}\,t\,(H_0+H_{\mathrm{int}})}\right) &= -\mathrm{i}\,e^{\mathrm{i}\,t\,H_0}\,H_{\mathrm{int}}\,e^{-\mathrm{i}\,t\,(H_0+H_{\mathrm{int}})} \\ &= -\mathrm{i}\,e^{\mathrm{i}\,t\,H_0}\,H_{\mathrm{int}}\,e^{-\mathrm{i}\,t\,H_0}\,e^{\mathrm{i}\,t\,H_0}\,e^{-\mathrm{i}\,t\,(H_0+H_{\mathrm{int}})} \\ &= -\mathrm{i}\,H_{\mathrm{int}}(t)\,e^{\mathrm{i}\,t\,H_0}\,e^{-\mathrm{i}\,t\,(H_0+H_{\mathrm{int}})} \quad ,\end{aligned} \tag{4.5}$$

where we would like to emphasize that the time evolution

$$H_{\mathrm{int}}(t) = e^{\mathrm{i}\,t\,H_0}\,H_{\mathrm{int}}\,e^{-\mathrm{i}\,t\,H_0} = \sum_{n\geq 3} \frac{\lambda_n}{n!} \int_{\mathbb{R}^{d-1}} \left(\Phi_0(x)\right)^n \mathrm{d}\mathbf{x} \tag{4.6}$$

with respect to the free Hamiltonian H_0 can be expressed in terms of the free Heisenberg picture field operator $\Phi_0(x)$. From the definition of the time-ordered product T in Section 3.4, it is easy to check that the unique solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} U(t,t_0) = -\mathrm{i}\, H_{\mathrm{int}}(t)\, U(t,t_0) \quad , \qquad U(t_0,t_0) = 1 \quad , \tag{4.7a}$$

for some arbitrary initial time $t_0 \in \mathbb{R}$, is given in the case where $t \geq t_0$ is later than the initial time by the time-ordered exponential

$$U(t,t_0) = \mathsf{T} \exp\Big(-\mathrm{i} \int_{t_0}^{t} H_{\mathrm{int}}(t')\, \mathrm{d}t' \Big) \qquad (\text{for } t \geq t_0) \quad , \tag{4.7b}$$

which is defined by the following series

$$\mathsf{T} \exp\Big(-\mathrm{i} \int_{t_0}^{t} H_{\mathrm{int}}(t')\, \mathrm{d}t' \Big) = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int_{t_0}^{t} \cdots \int_{t_0}^{t} \mathsf{T}\Big(H_{\mathrm{int}}(t_1) \cdots H_{\mathrm{int}}(t_n) \Big)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n \quad . \tag{4.7c}$$

In the case where $t \leq t_0$ is earlier than t_0, the solution $U(t,t_0)$ of (4.7a) can be expressed in terms of the adjoint

$$U(t,t_0) = U(t_0,t)^\dagger \qquad (\text{for } t \leq t_0) \quad , \tag{4.7d}$$

where the right-hand side is defined by the time-ordered exponential (4.7b) since $t_0 \geq t$. One can prove that this family of operators satisfies the following composition identity

$$U(t_2,t_1)\, U(t_1,t_0) = U(t_2,t_0) \tag{4.8}$$

for all $t_2, t_1, t_0 \in \mathbb{R}$, hence they behave like unitary time evolution operators. With these preparations, we can now express the interacting Heisenberg picture field operator $\Phi(x)$ from (4.4a) in terms of the free one $\Phi_0(x)$ as follows

$$\Phi(x) = U(0,t)\, \Phi_0(x)\, U(t,0) \quad , \tag{4.9}$$

where the argument 0 is due to our choice of initial time $t_0 = 0$ at which we have set up the Hamiltonian formalism. This formula shows that the new features arising from perturbative interactions can be encoded in the family of operators $U(t,t_0)$ from (4.7), which is constructed from data given by the underlying free QFT, namely the free quantum fields $\Phi_0(x)$, which enter the free time evolution of the interaction Hamiltonian (4.6), and their time-ordered products T.

It is important to emphasize that interactions do not only effect the field operators but also the vacuum state of the QFT. Indeed, the vacuum state $|0\rangle \in \mathcal{H}$ for the free QFT from Section 3.3 is not an eigenstate of the full Hamiltonian $H = H_0 + H_{\mathrm{int}}$, hence it cannot be the vacuum of the interacting QFT. The argument is quite simple: The interaction Hamiltonian H_{int} contains terms of the form Φ^n, for $n \geq 3$, which when expanded in the creation and annihilation operators leads to nonvanishing terms like $a^\dagger(\mathbf{k}_1) \cdots a^\dagger(\mathbf{k}_n)$ that consist only of creation operators. This means

that the eigenstate equation $H|0\rangle = H_0|0\rangle + H_{\text{int}}|0\rangle = E\,|0\rangle$ cannot hold true for any eigenvalue E, because the left-hand side consists of a superposition of multiparticle states while the right-hand side has particle number zero.

There exists a very useful and nice result, called the *Gell-Mann and Low Theorem*, that allows us to compute the vacuum state of the interacting theory $|\Omega\rangle \in \mathcal{H}$ in terms of the vacuum state of the free theory $|0\rangle \in \mathcal{H}$. The key idea is to assume that in the far future $t \to +\infty$ and in the far past $t \to -\infty$ the interactions can be neglected, which is technically implemented by a so-called adiabatic switching of the interaction Hamiltonian. This allows us to identify asymptotically in the far future/past the interacting theory with the free one. Using the operators (4.7), we then define the states

$$|\Omega^{\pm}\rangle := U(0,\pm\infty)|0\rangle \in \mathcal{H} \quad , \tag{4.10}$$

where again the argument 0 is due to our choice of initial time $t_0 = 0$ at which we have set up the Hamiltonian formalism. To interpret these states, it is useful to allow for a general initial time $t_0 \in \mathbb{R}$, for which they read as $|\Omega^{\pm}\rangle = U(t_0,\pm\infty)|0\rangle$. If we choose t_0 to be in the far future/past, i.e. $t_0 \to \pm\infty$, this simplifies to $|\Omega^{\pm}\rangle = U(\pm\infty,\pm\infty)|0\rangle = |0\rangle$. Hence, we obtain the following physical interpretation: $|\Omega^{\pm}\rangle$ is the state that coincides with the free vacuum state in the far future/past. Since the far future/past are often called the out/in regions, one says that $|\Omega^{+}\rangle$ is the *out vacuum* and $|\Omega^{-}\rangle$ is the *in vacuum*. With some analytical efforts and care (see e.g. here or Example 8.1 in the textbook by Greiner/Reinhardt), one can show that both states $|\Omega^{\pm}\rangle$ are eigenstates of the full Hamiltonian $H = H_0 + H_{\text{int}}$ with the same eigenvalue as the interacting vacuum state $|\Omega\rangle$. Assuming that the vacuum state is nondegenerate, it follows that all three states are the same, up to complex phases

$$|\Omega\rangle = e^{\mathrm{i}\alpha^{+}}|\Omega^{+}\rangle = e^{\mathrm{i}\alpha^{-}}|\Omega^{-}\rangle \quad . \tag{4.11}$$

The phase difference $e^{\mathrm{i}(\alpha^{+}-\alpha^{-})}$ can be computed from (4.10) and (4.7)

$$e^{\mathrm{i}(\alpha^{+}-\alpha^{-})} = e^{\mathrm{i}(\alpha^{+}-\alpha^{-})}\,\langle\Omega|\Omega\rangle = \langle\Omega^{+}|\Omega^{-}\rangle = \langle 0|U(+\infty,-\infty)|0\rangle \quad , \tag{4.12}$$

where we also used the standard normalization condition $\langle\Omega|\Omega\rangle = 1$ of the interacting vacuum.

Combining (4.9), (4.10) and (4.11), one obtains nice formulas to compute vacuum expectation values of operators in the interacting QFT in terms of vacuum expectation values in the free QFT. Let us illustrate this for the time-ordered n-point functions. Consider $\langle\Omega|\mathsf{T}(\Phi(x_1)\,\Phi(x_2)\cdots\Phi(x_n))|\Omega\rangle$ and assume without loss of generality that $x_1,\dots,x_n \in \mathbb{R}^d$ are already time-ordered. (This avoids notational clutter coming from the time-ordering permutation (3.55).) We can then write

$$\begin{aligned}
\langle\Omega|\mathsf{T}(\Phi(x_1)\,\Phi(x_2)\cdots\Phi(x_n))|\Omega\rangle &= \langle\Omega|\Phi(x_1)\,\Phi(x_2)\cdots\Phi(x_n)|\Omega\rangle \\
&= \langle\Omega|U(0,t_1)\,\Phi_0(x_1)\,U(t_1,t_2)\,\Phi_0(x_2)\,\cdots\Phi_0(x_n)\,U(t_n,0)|\Omega\rangle \\
&= e^{-\mathrm{i}(\alpha^{+}-\alpha^{-})}\,\langle\Omega^{+}|U(0,t_1)\,\Phi_0(x_1)\,U(t_1,t_2)\,\Phi_0(x_2)\,\cdots\Phi_0(x_n)\,U(t_n,0)|\Omega^{-}\rangle \\
&= \frac{\langle 0|U(+\infty,t_1)\,\Phi_0(x_1)\,U(t_1,t_2)\,\Phi_0(x_2)\,\cdots\Phi_0(x_n)\,U(t_n,-\infty)|0\rangle}{\langle 0|U(+\infty,-\infty)|0\rangle} \quad ,
\end{aligned} \tag{4.13}$$

where we frequently used the composition property of the U-operators (4.8). Observe that the expression in the last line comes naturally time-ordered, hence we can apply the time-ordering T without changing it. This allows us to rewrite this expression in the following more compact form

$$\begin{aligned} U(+\infty,t_1)\,\Phi_0(x_1)\,U(t_1,t_2)\,\Phi_0(x_2)\,\cdots\Phi_0(x_n)\,U(t_n,-\infty) \\ &= \mathsf{T}\Big(U(+\infty,t_1)\,\Phi_0(x_1)\,U(t_1,t_2)\,\Phi_0(x_2)\,\cdots\Phi_0(x_n)\,U(t_n,-\infty)\Big) \\ &= \mathsf{T}\Big(\Phi_0(x_1)\,\Phi_0(x_2)\,\cdots\Phi_0(x_n)\,U(+\infty,t_1)\,U(t_1,t_2)\,\cdots\,U(t_n,-\infty)\Big) \\ &= \mathsf{T}\Big(\Phi_0(x_1)\,\Phi_0(x_2)\,\cdots\Phi_0(x_n)\,U(+\infty,-\infty)\Big) \\ &= \mathsf{T}\Big(\Phi_0(x_1)\,\Phi_0(x_2)\,\cdots\Phi_0(x_n)\,e^{-\mathrm{i}\int_{-\infty}^{\infty}H_{\mathrm{int}}(t)\,\mathrm{d}t}\Big) \quad . \end{aligned} \tag{4.14}$$

Using that

$$-\int_{-\infty}^{\infty}H_{\mathrm{int}}(t)\,\mathrm{d}t = -\sum_{n\geq 3}\frac{\lambda_n}{n!}\int_{\mathbb{R}^d}\big(\Phi_0(x)\big)^n\,\mathrm{d}x = S_{\mathrm{int}}[\Phi_0] \tag{4.15}$$

is simply the interaction term of the action functional, evaluated on the free quantum field Φ_0, we can write the result in the following neater way

$$\langle\Omega|\mathsf{T}\big(\Phi(x_1)\cdots\Phi(x_n)\big)|\Omega\rangle = \frac{\langle 0|\mathsf{T}\Big(\Phi_0(x_1)\cdots\Phi_0(x_n)\,e^{\mathrm{i}\,S_{\mathrm{int}}[\Phi_0]}\Big)|0\rangle}{\langle 0|\mathsf{T}\Big(e^{\mathrm{i}\,S_{\mathrm{int}}[\Phi_0]}\Big)|0\rangle} \quad . \tag{4.16}$$

This is called the *Gell-Mann and Low reduction formula* and it is one of the most important formulas in QFT. Indeed, this formula achieves something truly amazing: It reduces the problem of computing time-ordered expectation values of *interacting* field operators $\Phi(x)$ in the *interacting* vacuum state $|\Omega\rangle$ to computing time-ordered expectation values of *free* field operators $\Phi_0(x)$ in the *free* vacuum state $|0\rangle$.

4.2 Wick's theorem

Evaluating the right-hand side of the Gell-Mann and Low reduction formula (4.16) turns out to be rather cumbersome. One has to compute a series of time-ordered products

$$\langle 0|\mathsf{T}\Big(\Phi_0(x_1)\cdots\Phi_0(x_n)\,e^{\mathrm{i}\,S_{\mathrm{int}}[\Phi_0]}\Big)|0\rangle = \sum_{m\geq 0}\frac{\mathrm{i}^m}{m!}\langle 0|\mathsf{T}\Big(\Phi_0(x_1)\cdots\Phi_0(x_n)\,\big(S_{\mathrm{int}}[\Phi_0]\big)^m\Big)|0\rangle \quad , \tag{4.17}$$

where we recall that the interaction term $S_{\mathrm{int}}[\Phi_0]$ is a polynomial in the free quantum fields Φ_0 and hence it adds to the number of fields under the time-ordered product. This means that, even for computing the *interacting* time-ordered n-point functions $\langle\Omega|\mathsf{T}\big(\Phi(x_1)\cdots\Phi(x_n)\big)|\Omega\rangle$ for a very low n, we have to know the *free* time-ordered n-point functions for a much larger n, which is determined by the form of the interaction and by the order m of the perturbative expansion we wish to reach.

For example, if we take $S_{\text{int}}[\Phi_0] = -\frac{\lambda}{4!}\int_{\mathbb{R}^d}\big(\Phi_0(x)\big)^4\,\mathrm{d}x$ to be a quartic interaction and want to determine the interacting time-ordered 2-point function to order λ^2, we have to compute

$$\begin{aligned}\langle 0|\mathsf{T}\Big(\Phi_0(x_1)\,\Phi_0(x_2)\,e^{\mathrm{i}\,S_{\text{int}}[\Phi_0]}\Big)|0\rangle &= \langle 0|\mathsf{T}\Big(\Phi_0(x_1)\,\Phi_0(x_2)\Big)|0\rangle \\ &+ \frac{-\mathrm{i}\,\lambda}{4!}\int_{\mathbb{R}^d}\langle 0|\mathsf{T}\Big(\Phi_0(x_1)\,\Phi_0(x_2)\,\big(\Phi_0(y)\big)^4\Big)|0\rangle\,\mathrm{d}y \\ &+ \frac{1}{2!}\left(\frac{-\mathrm{i}\,\lambda}{4!}\right)^2\int_{\mathbb{R}^{2d}}\langle 0|\mathsf{T}\Big(\Phi_0(x_1)\,\Phi_0(x_2)\,\big(\Phi_0(y_1)\big)^4\,\big(\Phi_0(y_2)\big)^4\Big)|0\rangle\;\mathrm{d}y_1\,\mathrm{d}y_2 + \mathcal{O}(\lambda^3) \quad . \end{aligned} \tag{4.18}$$

This involves the 2, 6 and 10-point functions, and when we would like to know the order λ^3 contribution then we also need the 14-point function. (I think the pattern becomes clear, isn't it?) So what we need are techniques to determine the time-ordered n-point functions $\langle 0|\mathsf{T}(\Phi_0(x_1)\cdots\Phi_0(x_n))|0\rangle$ of the free theory for arbitrary n.

A suitable technique for this task is provided by *Wick's theorem*, which allows us to rewrite the time-ordered product $\mathsf{T}\big(\Phi_0(x_1)\cdots\Phi_0(x_n)\big)$ as a sum of normal ordered products (see Definition 3.1) and contractions given by the Feynman propagator. Before we state and prove Wick's theorem in its full form, let us explore this problem for low n in order to get some intuition. For $n = 1$, we have the obvious identity

$$\mathsf{T}\big(\Phi_0(x_1)\big) = :\Phi_0(x_1): = \Phi_0(x_1) \quad , \tag{4.19}$$

which tells us that both the time-ordered product and the normal ordering of $\Phi_0(x_1)$ agree with the field operator $\Phi_0(x_1)$ itself. Consider now the case $n = 2$. Recalling the expression (4.4b) of the free field operator in terms of annihilation and creation operators, let us write $\Phi_0(x) = a(x) + a^\dagger(x)$ to denote its decomposition into the annihilating and the creating part. Explicitly,

$$a(x) := \int_{\mathbb{R}^{d-1}}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\,a(\mathbf{k})\,e^{\mathrm{i}\,k\,x}\,\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \quad a^\dagger(x) := \int_{\mathbb{R}^{d-1}}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\,a^\dagger(\mathbf{k})\,e^{-\mathrm{i}\,k\,x}\,\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad . \tag{4.20}$$

For $x_1^0 \geq x_2^0$, we can then write

$$\begin{aligned}\mathsf{T}\big(\Phi_0(x_1)\,\Phi_0(x_2)\big) &= \Phi_0(x_1)\,\Phi_0(x_2) \\ &= a(x_1)\,a(x_2) + a(x_1)\,a^\dagger(x_2) + a^\dagger(x_1)\,a(x_2) + a^\dagger(x_1)\,a^\dagger(x_2) \\ &= a(x_1)\,a(x_2) + \big[a(x_1), a^\dagger(x_2)\big] + a^\dagger(x_2)\,a(x_1) + a^\dagger(x_1)\,a(x_2) + a^\dagger(x_1)\,a^\dagger(x_2) \\ &= :\Phi_0(x_1)\,\Phi_0(x_2): + \big[a(x_1), a^\dagger(x_2)\big] = :\Phi_0(x_1)\,\Phi_0(x_2): + W_2(x_1, x_2) \quad , \end{aligned} \tag{4.21a}$$

where W_2 is the 2-point Wightman function (3.49b), while for $x_2^0 \geq x_1^0$ we find

$$\mathsf{T}\big(\Phi_0(x_1)\,\Phi_0(x_2)\big) = :\Phi_0(x_2)\,\Phi_0(x_1): + W_2(x_2, x_1) \quad . \tag{4.21b}$$

Since the normal ordering $:\Phi_0(x_1)\,\Phi_0(x_2): = :\Phi_0(x_2)\,\Phi_0(x_1):$ is symmetric under the exchange of x_1 and x_2, we find the following case-independent identity

$$\mathsf{T}\big(\Phi_0(x_1)\,\Phi_0(x_2)\big) = :\Phi_0(x_1)\,\Phi_0(x_2): + \Delta_F(x_1 - x_2) \quad , \tag{4.22}$$

where Δ_F is the Feynman propagator (3.56b). Introducing the shorthand notation

$$\Phi_i \;:=\; \Phi_0(x_i) \quad , \quad (\Delta_F)_{ij} \;:=\; \Delta_F(x_i - x_j) \quad , \tag{4.23}$$

we can write this more compactly as

$$\mathsf{T}(\Phi_1\,\Phi_2) = :\Phi_1\,\Phi_2: + (\Delta_F)_{12} \quad . \tag{4.24}$$

Playing the same game for $n = 3$, you will find that

$$\mathsf{T}(\Phi_1\,\Phi_2\,\Phi_3) = :\Phi_1\,\Phi_2\,\Phi_3: + (\Delta_F)_{12}\,:\Phi_3: + (\Delta_F)_{13}\,:\Phi_2: + (\Delta_F)_{23}\,:\Phi_1: \quad , \tag{4.25}$$

and for $n = 4$ you will find that

$$\begin{aligned}\mathsf{T}(\Phi_1\,\Phi_2\,\Phi_3\,\Phi_4) = {}& :\Phi_1\,\Phi_2\,\Phi_3\,\Phi_4: + (\Delta_F)_{12}\,:\Phi_3\,\Phi_4: + (\Delta_F)_{13}\,:\Phi_2\,\Phi_4: + (\Delta_F)_{14}\,:\Phi_2\,\Phi_3: \\ & + (\Delta_F)_{23}\,:\Phi_1\,\Phi_4: + (\Delta_F)_{24}\,:\Phi_1\,\Phi_3: + (\Delta_F)_{34}\,:\Phi_1\,\Phi_2: \\ & + (\Delta_F)_{12}\,(\Delta_F)_{34} + (\Delta_F)_{13}\,(\Delta_F)_{24} + (\Delta_F)_{14}\,(\Delta_F)_{23} \quad .\end{aligned} \tag{4.26}$$

Observe that there emerges the following pattern: The time-ordered product of n field operators seems to be equal to the normal ordering of the field operators, plus all possible contractions with the Feynman propagator, plus all possible double contractions with the Feynman propagator, and so on. This is indeed the case. To give a precise statement and proof of this result, we need some additional notation. Given two positive integers n and q, we denote by $\mathcal{P}_n^q$ the set of all collections $\{\{i_1, j_1\}, \ldots, \{i_q, j_q\}\}$ of q pairs of elements in $I_n := \{1, \ldots, n\}$, such that any of the $2q$ elements $i_1, \ldots, i_q, j_1, \ldots, j_q \in I_n$ are distinct. By convention, we set $\mathcal{P}_n^0$ to be the set with a single element given by the empty collection $\{\}$. Note that $\mathcal{P}_n^q = \emptyset$ whenever $2q > n$ because in this case there are not enough elements in $I_n = \{1, \ldots, n\}$ for the $2q$ elements to be distinct. We denote elements of $\mathcal{P}_n^q$ by the shorthand notation $\underline{\{i,j\}} := \{\{i_1, j_1\}, \ldots, \{i_q, j_q\}\}$ and write $I_n \setminus \underline{\{i,j\}} := I_n \setminus \bigcup_{l=1}^q \{i_l, j_l\}$ for the complement in $I_n = \{1, \ldots, n\}$ of all the pairs.

Theorem 4.1 (Wick's theorem for time-ordered products)**.** *With the notations established above, the following identity holds true for all positive integers n*

$$\mathsf{T}\Big(\prod_{k=1}^{n} \Phi_k\Big) = \sum_{q=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\underline{\{i,j\}} \in \mathcal{P}_n^q} \prod_{l=1}^{q} (\Delta_F)_{i_l j_l} \; : \prod_{k \in I_n \setminus \underline{\{i,j\}}} \Phi_k: \quad , \tag{4.27}$$

where we recall that $\lfloor \frac{n}{2} \rfloor$ denotes the greatest integer less than or equal to $\frac{n}{2}$.

Remark 4.2. I have to admit that this formula looks a bit heavy, but I don't see any better way to write it down in a closed form. What is usually presented in the physics literature is a less precise, but easier to read and digest, formula of the form

$$\begin{aligned}\mathsf{T}\Big(\Phi_1\,\Phi_2\,\Phi_3 \cdots \Phi_n\Big) = {}& :\Phi_1\,\Phi_2\,\Phi_3 \cdots \Phi_n: \\ & + :\overbrace{\Phi_1\,\Phi_2}\,\Phi_3 \cdots \Phi_n: + :\overbrace{\Phi_1\,\Phi_2\,\Phi_3} \cdots \Phi_n: + \cdots + :\overbrace{\Phi_1\,\Phi_2\,\Phi_3 \cdots \Phi_n}: \\ & + :\overbrace{\Phi_1}\,\overbrace{\Phi_2}\,\overbrace{\Phi_3}\,\overbrace{\Phi_4} \cdots \Phi_n: + :\overbrace{\Phi_1\,\Phi_2\,\Phi_3\,\Phi_4} \cdots \Phi_n: + \cdots + :\Phi_1 \cdots \overbrace{\Phi_{n-3}\,\Phi_{n-2}}\,\overbrace{\Phi_{n-1}\,\Phi_n}: \\ & + \quad \text{higher contractions} \quad ,\end{aligned} \tag{4.28}$$

where the contraction symbols mean insertions of Feynman propagators, i.e. $\overbrace{\Phi_i\,\Phi_j} = (\Delta_F)_{ij}$. By direct inspection you can convince yourself that this formula is the same as (4.27).

Proof. We have seen above that (4.27) holds true for $n = 1, 2, 3, 4$, see (4.19), (4.24), (4.25) and (4.26). We prove the general result by induction. Suppose that (4.27) holds true for n. Consider $n+1$ fields $\Phi_i = \Phi_0(x_i)$ at the positions $x_1, \dots, x_{n+1} \in \mathbb{R}^d$ that we may assume without loss of generality to be time-ordered. We can then write

$$\mathsf{T}\Big(\prod_{k=1}^{n+1} \Phi_k\Big) = \mathsf{T}\Big(\prod_{k=1}^{n} \Phi_k\Big)\, \Phi_{n+1} = \Big(\sum_{q=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\underline{\{i,j\}} \in \mathcal{P}_n^q} \prod_{l=1}^{q} (\Delta_F)_{i_l j_l} \;:\prod_{k \in I_n \setminus \underline{\{i,j\}}} \Phi_k : \Big)\, \Phi_{n+1} \quad . \tag{4.29}$$

Decomposing $\Phi_{n+1} = a_{n+1} + a_{n+1}^\dagger$ into the annihilating and creating parts (4.20), we compute

$$\begin{aligned} :\prod_{k \in I_n \setminus \underline{\{i,j\}}} \Phi_k :\, \Phi_{n+1} &= :\prod_{k \in I_n \setminus \underline{\{i,j\}}} \Phi_k : a_{n+1} \;+\; a_{n+1}^\dagger :\prod_{k \in I_n \setminus \underline{\{i,j\}}} \Phi_k : \;+\; \Big[:\prod_{k \in I_n \setminus \underline{\{i,j\}}} \Phi_k :, a_{n+1}^\dagger \Big] \\ &= :\prod_{k \in I_{n+1} \setminus \underline{\{i,j\}}} \Phi_k : \;+\; \sum_{r \in I_n \setminus \underline{\{i,j\}}} (\Delta_F)_{r(n+1)} :\prod_{\substack{k \in I_n \setminus \underline{\{i,j\}} \\ k \neq r}} \Phi_k : \quad . \end{aligned} \tag{4.30}$$

To obtain the first term of the second line, we have used that the first two terms of the first line correspond to including Φ_{n+1} into the normal ordered product. (Note the I_{n+1} instead of I_n.) To obtain the second term of the second line, we have used the Leibniz rule to evaluate the commutator. Inserting this result back into (4.29), we get

$$\begin{aligned} \mathsf{T}\Big(\prod_{k=1}^{n+1} \Phi_k\Big) &= \sum_{q=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\underline{\{i,j\}} \in \mathcal{P}_n^q} \prod_{l=1}^{q} (\Delta_F)_{i_l j_l} \;:\prod_{k \in I_{n+1} \setminus \underline{\{i,j\}}} \Phi_k : \\ &\qquad + \sum_{q=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\underline{\{i,j\}} \in \mathcal{P}_n^q} \sum_{r \in I_n \setminus \underline{\{i,j\}}} \Big(\prod_{l=1}^{q} (\Delta_F)_{i_l j_l} \Big) (\Delta_F)_{r(n+1)} :\prod_{\substack{k \in I_n \setminus \underline{\{i,j\}} \\ k \neq r}} \Phi_k : \\ &= \sum_{q=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{\underline{\{i,j\}} \in \mathcal{P}_{n+1}^q} \prod_{l=1}^{q} (\Delta_F)_{i_l j_l} \;:\prod_{k \in I_{n+1} \setminus \underline{\{i,j\}}} \Phi_k : \quad . \end{aligned} \tag{4.31}$$

To understand the last equality, note that an element $\underline{\{i,j\}} = \{\{i_1, j_1\}, \dots, \{i_q, j_q\}\} \in \mathcal{P}_{n+1}^q$ either does or does not include $n+1$. If it does not include $n+1$, then the corresponding term is part of the first line, and if it does include $n+1$ then it is part of the second line. So all terms match up in the last equality and we have completed the proof. □

Taking the vacuum expectation value of the time-ordered products in (4.27) simplifies considerably the expression on the right-hand side because $\langle 0| :\Phi_1 \cdots \Phi_k: |0\rangle = 0$ for any number $k \geq 1$ of field operators. This is because, by Definition 3.1, the operator $:\Phi_1 \cdots \Phi_k:$ is such that all annihilation operators are to the right and all creation operators are to the left, hence each summand vanishes using either $a(\mathbf{k})|0\rangle = 0$ or its adjoint $\langle 0| a^\dagger(\mathbf{k}) = 0$. Let us state this observation as a corollary.

Corollary 4.3 (Wick's theorem for time-ordered n-point functions)**.** *The free time-ordered $2n+1$-point function for an odd number of field operators vanishes*

$$\langle 0|\mathsf{T}\Big(\prod_{k=1}^{2n+1}\Phi_k\Big)|0\rangle = 0 \quad . \tag{4.32a}$$

The free time-ordered $2n$-point function for an even number of field operators is given by

$$\langle 0|\mathsf{T}\Big(\prod_{k=1}^{2n}\Phi_k\Big)|0\rangle = \sum_{\underline{\{i,j\}}\in\mathcal{P}^n_{2n}}\prod_{l=1}^{n}(\Delta_F)_{i_l j_l} \quad , \tag{4.32b}$$

where we recall that $\mathcal{P}^n_{2n}$ denotes the set of all collections $\{\{i_1,j_1\},\dots,\{i_n,j_n\}\}$ of n pairs of elements in $I_{2n}=\{1,\dots,2n\}$, such that any of the $2n$ elements $i_1,\dots,i_n,j_1,\dots,j_n\in I_{2n}$ are distinct. Note that $\mathcal{P}^n_{2n}$ is precisely the set of partitions of the set $I_{2n}=\{1,\dots,2n\}$ into pairs.

Example 4.4. The first few time-ordered n-point functions read as follows

$$\begin{aligned}
\langle 0|\mathsf{T}(\Phi_1)|0\rangle &= 0 \quad ,\\
\langle 0|\mathsf{T}(\Phi_1\,\Phi_2)|0\rangle &= (\Delta_F)_{12} \quad ,\\
\langle 0|\mathsf{T}(\Phi_1\,\Phi_2\,\Phi_3)|0\rangle &= 0 \quad ,\\
\langle 0|\mathsf{T}(\Phi_1\,\Phi_2\,\Phi_3\,\Phi_4)|0\rangle &= (\Delta_F)_{12}\,(\Delta_F)_{34}+(\Delta_F)_{13}\,(\Delta_F)_{24}+(\Delta_F)_{14}\,(\Delta_F)_{23} \quad ,\\
\langle 0|\mathsf{T}(\Phi_1\,\Phi_2\,\Phi_3\,\Phi_4\,\Phi_5)|0\rangle &= 0 \quad .
\end{aligned} \tag{4.33}$$

The higher time-ordered $2n$-point functions are considerably more involved, because there exist $\frac{(2n)!}{2^n\,n!}$ many partitions of $\{1,\dots,2n\}$ into pairs. This corresponds to $\frac{(2n)!}{2^n\,n!}$ many terms in the sum (4.32), which quickly becomes very large. For instance, for the $6=2\times 3$-point function, there are $\frac{6!}{2^3\,3!}=15$ many terms, and for the $8=2\times 4$-point function, there are $\frac{8!}{2^4\,4!}=105$ many terms. This high complexity suggests that it would be a good idea to use computer assistance in order to evaluate Wick's theorem, which is indeed what QFT practitioners are doing.

4.3 Interacting Feynman propagator

In order to get a better understanding of Wick's theorem, let us compute the lowest order corrections to the interacting Feynman propagator

$$\langle\Omega|\mathsf{T}\big(\Phi(x_1)\,\Phi(x_2)\big)|\Omega\rangle = \frac{\langle 0|\mathsf{T}\Big(\Phi_0(x_1)\,\Phi_0(x_2)\,e^{\mathrm{i}\,S_{\mathrm{int}}[\Phi_0]}\Big)|0\rangle}{\langle 0|\mathsf{T}\Big(e^{\mathrm{i}\,S_{\mathrm{int}}[\Phi_0]}\Big)|0\rangle} \quad . \tag{4.34}$$

For this we have to choose an interaction term S_{int} for the action functional. We consider the following two cases that are not only motivated by their simplicity, but also by their similarity to the kind of interactions that appear in the standard model of particle physics:

- A cubic interaction term $S_{\mathrm{int}}[\Phi_0]=-\frac{\lambda}{3!}\int_{\mathbb{R}^d}\big(\Phi_0(x)\big)^3\,\mathrm{d}x$, which is called Φ^3-*theory*.
- A quartic interaction term $S_{\mathrm{int}}[\Phi_0]=-\frac{\lambda}{4!}\int_{\mathbb{R}^d}\big(\Phi_0(x)\big)^4\,\mathrm{d}x$, which is called Φ^4-*theory*.

Of course, our formalism works for any interaction term $S_{\mathrm{int}}[\Phi_0]=-\sum_{n\geq 3}\frac{\lambda_n}{n!}\int_{\mathbb{R}^d}\big(\Phi_0(x)\big)^n\,\mathrm{d}x$, but working at this level of generality would unnecessarily complicate our studies.

Φ^4-theory:

Let us start by looking at the numerator of (4.34) and expand it to the first order in the coupling constant λ. Using a similar shorthand notation as in Section 4.2, we write $\Phi_1 := \Phi_0(x_1)$, $\Phi_2 := \Phi_0(x_2)$ and $\Phi_y := \Phi_0(y)$. We obtain

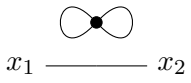
$$\langle 0|\mathsf{T}\Big(\Phi_1\,\Phi_2\,e^{\mathrm{i}\,S_{\mathrm{int}}[\Phi_0]}\Big)|0\rangle = \langle 0|\mathsf{T}(\Phi_1\,\Phi_2)|0\rangle - \frac{\mathrm{i}\,\lambda}{4!}\int_{\mathbb{R}^d}\langle 0|\mathsf{T}\Big(\Phi_1\,\Phi_2\,(\Phi_y)^4\Big)|0\rangle\,\mathrm{d}y + \mathcal{O}(\lambda^2) \quad . \tag{4.35}$$

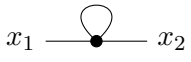
We can evaluate these expressions by using Wick's theorem, where the variant from Corollary 4.3 is particularly useful. The first term becomes a Feynman propagator $\langle 0|\mathsf{T}(\Phi_1\,\Phi_2)|0\rangle = (\Delta_F)_{12}$ and the second term yields a sum over all possible contractions of pairs according to (4.32). The latter contractions come in two different types:

Type 1: $(\Delta_F)_{12}\,(\Delta_F)_{yy}\,(\Delta_F)_{yy}$ is obtained by contracting Φ_1 with Φ_2 and the four Φ_y among themselves.

Type 2: $(\Delta_F)_{1y}\,(\Delta_F)_{2y}\,(\Delta_F)_{yy}$ is obtained by contracting Φ_1 with one of the Φ_y, Φ_2 with another one of the Φ_y, and the remaining two Φ_y among themselves.

We can visualize these two types of contributions by the following *Feynman diagrams* that keep track of who is contracted against who:

Type 1: 

Type 2: 

The lines represent free Feynman propagators and the vertices represent the interaction term, which explains why they are 4-valent for Φ^4-theory. The type 1 diagram can be interpreted physically as follows: While a particle propagates from x_1 to x_2, the vacuum undergoes a quantum fluctuation that creates and annihilates virtual particles. Such vacuum fluctuations that are not connected to any of the external points x_i are often called *vacuum bubbles*, which makes sense given the shape of the diagram. The interpretation of the type 2 diagram is as follows: A particle propagates from x_1 and then transforms at some point y into three particles of which two annihilate each other and the third propagates to x_2. This is also a quantum fluctuation, but in contrast to vacuum bubbles it is connected to the external points x_i.

We still have to count the multiplicities of these two types of contributions, i.e. we have to figure out how often they appear in the sum (4.32). The associated combinatorial problem is to count in how many ways the set $\{1, 2, y, y, y, y\}$ can be partitioned into $\{\{1, 2\}, \{y, y\}, \{y, y\}\}$ (for type 1) and into $\{\{1, y\}, \{2, y\}, \{y, y\}\}$ (for type 2).

Type 1: There is 1 possibility to pick $\{1, 2\}$. Fixing any y, there are 3 possibilities to pick its partner from the remaining three y's to form a pair $\{y, y\}$. The remaining pair $\{y, y\}$ is fixed by these previous choices. Hence, there are 3 contributions to type 1.

Type 2: There are 4 possibilities to pick a y for $\{1, y\}$. Once this has been picked, we are left with 3 possibilities to pick a y for $\{2, y\}$. The remaining pair $\{y, y\}$ is fixed by these previous choices. Hence, there are $4 \times 3 = 12$ contributions to type 2.

With these preparations, we can now finally write down an expression for (4.35)

$$\begin{aligned}\langle 0|\mathsf{T}\Big(\Phi_0(x_1)\,\Phi_0(x_2)\,e^{\mathrm{i}\,S_{\mathrm{int}}[\Phi_0]}\Big)|0\rangle = \Delta_F(x_1-x_2) - \frac{\mathrm{i}\,\lambda}{8}\int_{\mathbb{R}^d}\Delta_F(x_1-x_2)\,\big(\Delta_F(0)\big)^2\,\mathrm{d}y \\ - \frac{\mathrm{i}\,\lambda}{2}\int_{\mathbb{R}^d}\Delta_F(x_1-y)\,\Delta_F(x_2-y)\,\Delta_F(0)\,\mathrm{d}y \;+\; \mathcal{O}(\lambda^2) \quad ,\end{aligned} \tag{4.36a}$$

which we can also display graphically as

$$\langle 0|\mathsf{T}\Big(\Phi_0(x_1)\,\Phi_0(x_2)\,e^{\mathrm{i}\,S_{\mathrm{int}}[\Phi_0]}\Big)|0\rangle \;=\; \text{——} \;+\; \frac{1}{8}\;\text{⟲——}\;+\;\frac{1}{2}\;\text{—⟲—}\;+\;\mathcal{O}(\lambda^2) \quad . \tag{4.36b}$$

The prefactors in this graphical representation are called the *symmetry factors* and they are given by the inverse of the order of the symmetry group of the diagram that fixes the external points x_i. In our example, the vacuum bubble diagram has a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry given by 1.) flipping the left with the right bubble, 2.) flipping the two legs of the left bubble and 3.) flipping the two legs of the right bubble. This is a group of order $2 \times 2 \times 2 = 8$, hence the symmetry factor is $\frac{1}{8}$. The third diagram has only a $\mathbb{Z}_2$-symmetry given by flipping the two legs of the loop, hence the symmetry factor is $\frac{1}{2}$.

By direct inspection, one sees that there is a simple way to translate between the graphical description and the explicit formula. This translation dictionary is given by the so-called *Feynman rules*, which for our example of Φ^4-theory read as follows:

- To every line between two points x and y assign a Feynman propagator

$$x\text{——}y \;=\; \Delta_F(x-y) \quad . \tag{4.37}$$

- To every vertex at a point z assign an integral

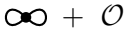
$$\text{×}_z \;=\; -\mathrm{i}\,\lambda\int_{\mathbb{R}^d}\mathrm{d}z \quad . \tag{4.38}$$

Warning 4.5. Looking more closely at the result (4.36), one immediately sees that both correction terms to the Feynman propagator are divergent. Note that there are two different origins for these divergences: First, the integrand for the vacuum bubble correction does not depend on y, hence it can be pulled out of the integral, leaving us with $\int_{\mathbb{R}^d}\mathrm{d}y = \infty$. This type of divergence is called an *infrared divergence* because it arises from a large length scale, namely the infinite volume of spacetime. Second, the free Feynman propagator $\Delta_F(0)$ at 0 can be written with the help of our Fourier transform formula (3.57) as

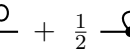
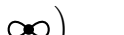
$$\Delta_F(0) = \int_{\mathbb{R}^d}\frac{-\mathrm{i}}{k^2+m^2-\mathrm{i}\,\epsilon}\,\frac{\mathrm{d}k}{(2\pi)^d} \quad , \tag{4.39}$$

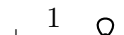
where here and in the following we suppress writing the limit $\lim_{\epsilon\to 0}$ to ease our notations. Even though we do not know yet how to evaluate such integrals (this will be explained in a later part of the module, see Chapter 8), this integral looks suspiciously divergent: We have d powers of k in the numerator via the integration measure $\mathrm{d}k = \mathrm{d}k^0 \cdots \mathrm{d}k^{d-1}$ and only 2 powers of k in the denominator, so the integrand will not fall off sufficiently fast for large k. This indeed turns out to be true and one finds that $\Delta_F(0)$ is divergent for all spacetime dimensions $d = 2, 3, 4, \ldots$. This type of divergence is called an *ultraviolet divergence* because it arises from large momenta, i.e. small length scales. For the moment, let's not worry too much about these divergent expression. We will learn later in Chapter 8 how to "cure" such divergences via suitable renormalization techniques.

To complete the description of the interacting Feynman propagator (4.34) to first order in the coupling constant λ, we still have to understand the denominator. Using the same methods as above, one finds in our graphical notation that

$$\langle 0|\mathsf{T}\big(e^{\mathrm{i}\,S_{\mathrm{int}}[\Phi_0]}\big)|0\rangle = 1 + \frac{1}{8}$$  $$+\ \mathcal{O}(\lambda^2) \quad . \tag{4.40}$$

Applying now a first order Taylor expansion in λ to the ratio in (4.34), we find

$$\langle\Omega|\mathsf{T}\big(\Phi(x_1)\,\Phi(x_2)\big)|\Omega\rangle \;=\; \frac{\big(\;\;+\;\tfrac{1}{8}$$  $$+\;\tfrac{1}{2}$$  $$\big)}{\big(1+\;\tfrac{1}{8}$$  $$\big)} \;+\; \mathcal{O}(\lambda^2)$$

$$= \;\;+\;\frac{1}{8}$$  $$+\;\frac{1}{2}$$ $$-\;\frac{1}{8}$$ $$+\;\mathcal{O}(\lambda^2)$$

$$= \;\;+\;\frac{1}{2}$$ $$+\;\mathcal{O}(\lambda^2) \quad . \tag{4.41}$$

Observation 4.6. Our calculation in (4.41) shows that the vacuum bubble contributions to the interacting Feynman propagator $\langle\Omega|\mathsf{T}\big(\Phi(x_1)\,\Phi(x_2)\big)|\Omega\rangle$ that come from the numerator and the denominator precisely cancel each other. This is not a coincidence! As shown in Example 8.3 in the textbook by Greiner/Reinhardt, for *any* choice of interaction term S_{int} and *any* interacting time-ordered n-point function $\langle\Omega|\mathsf{T}\big(\Phi(x_1)\cdots\Phi(x_n)\big)|\Omega\rangle$, the vacuum bubble contributions to the numerator of (4.16) cancel precisely the denominator. This allows us to introduce the following useful rule:

> **!!!** For computing the interacting time-ordered n-point functions $\langle\Omega|\mathsf{T}\big(\Phi(x_1)\cdots\Phi(x_n)\big)|\Omega\rangle$ in (4.16), one can ignore the denominator and all contributions to the numerator that come with vacuum bubbles.

You may now ask yourself: Is there any physical meaning and interpretation of the corrections (4.41) to the Feynman propagator? (At least qualitatively, because we haven't yet renormalized the divergences that we have discovered in Warning 4.5.) There is indeed! Recall from (3.57) that the Fourier transform

$$\widetilde{\Delta}_F(k) \;:=\; \int_{\mathbb{R}^d} \Delta_F(x)\, e^{-\mathrm{i}\,k\,x}\,\mathrm{d}x \;=\; \frac{-\mathrm{i}}{k^2 + m^2 - \mathrm{i}\,\epsilon} \tag{4.42}$$

of the free Feynman propagator has a pole at $k^2 = -m^2 + \mathrm{i}\,\epsilon$. Recall also that the parameter m^2 from the classical action functional could be interpreted as the mass of the associated particles. Taking the Fourier transform of the interacting Feynman propagator (4.41), we find

$$\begin{aligned}
\widetilde{\Delta}_F^{\mathrm{int}}(k) &= \widetilde{\Delta}_F(k) - \frac{\mathrm{i}\,\lambda}{2}\,\Delta_F(0)\,\big(\widetilde{\Delta}_F(k)\big)^2 + \mathcal{O}(\lambda^2) \\
&= \widetilde{\Delta}_F(k)\,\Big(1 - \frac{\mathrm{i}\,\lambda}{2}\,\Delta_F(0)\,\widetilde{\Delta}_F(k)\Big) + \mathcal{O}(\lambda^2) \\
&= \widetilde{\Delta}_F(k)\,\sum_{n=0}^{\infty}\Big(-\frac{\mathrm{i}\,\lambda}{2}\,\Delta_F(0)\,\widetilde{\Delta}_F(k)\Big)^n + \mathcal{O}(\lambda^2) \\
&= \frac{\widetilde{\Delta}_F(k)}{1 + \frac{\mathrm{i}\lambda}{2}\,\Delta_F(0)\,\widetilde{\Delta}_F(k)} + \mathcal{O}(\lambda^2) \\
&= \frac{-\mathrm{i}}{k^2 + m^2 + \frac{\lambda}{2}\,\Delta_F(0) - \mathrm{i}\,\epsilon} + \mathcal{O}(\lambda^2) \quad ,
\end{aligned} \tag{4.43}$$

where in the step from the second to the third line we have extended the term in the parenthesis to a geometric series, which is of course the same up to $\mathcal{O}(\lambda^2)$.

Observation 4.7. The calculation in (4.43) is very interesting: It shows that the particles associated with the interacting Φ^4-theory acquire a mass

$$m^2_{\mathrm{int}} := m^2 + \frac{\lambda}{2}\,\Delta_F(0) + \mathcal{O}(\lambda^2) \tag{4.44}$$

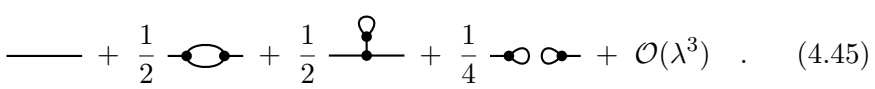
that is different from the particles associated with the underlying free QFT, whose mass m^2 is the parameter from the classical action functional. The origin of this mass shift is the loop diagram in (4.41), and of course there will be additional contributions when we go to higher order corrections in the coupling constant λ. This observation already gives some hints how one could try cure the divergences from Warning 4.5: Choose the free mass m^2 such that m^2_{int} gives the finite and physically desired value for the mass of the particle. This rough idea will be formalized later when we talk about renormalization.

Φ^3-theory:

Computing the interacting Feynman propagator (4.34) for Φ^3-theory works similarly as in the case of Φ^4-theory above. Hence, I will only spell out the relevant diagrams that contribute to this calculation and summarize the final result. I would recommend you to carry out these computations on your own to see if you get the details right.

Note that the first nontrivial correction to the Feynman propagator for Φ^3-theory is of order λ^2, which is due to the fact that all odd time-ordered n-point functions vanish by Corollary 4.3, hence in particular $\langle 0|\mathsf{T}(\Phi_1\,\Phi_2\,(\Phi_y)^3)|0\rangle = 0$. Recalling also from Observation 4.6 that all vacuum bubbles can be neglected, we find the following result

$\langle\Omega|\mathsf{T}(\Phi(x_1)\,\Phi(x_2))|\Omega\rangle$ =



$+ \mathcal{O}(\lambda^3)$. (4.45)

Note that the last term is the square of a contribution that already arises for the interacting 1-point function

$$\langle\Omega|\mathsf{T}(\Phi(x))|\Omega\rangle \;=\; \langle\Omega|\Phi(x)|\Omega\rangle \;=\; \frac{1}{2}$$ $$+\; \mathcal{O}(\lambda^2) \quad , \tag{4.46}$$

which describes the vacuum expectation value of the interacting field operator $\Phi(x)$. Whenever the latter is nonzero, it makes sense to redefine the field operator $\underline{\Phi}(x) := \Phi(x) - \langle\Omega|\Phi(x)|\Omega\rangle$ by subtracting its vacuum expectation value, which then describes the quantum fluctuation around $\langle\Omega|\Phi(x)|\Omega\rangle$. At the level of the Feynman propagator, this subtraction amounts to considering the interacting *connected Feynman propagator*

$$\begin{aligned}\langle\Omega|\mathsf{T}\big(\Phi(x_1)\,\Phi(x_2)\big)|\Omega\rangle_{\mathrm{c}} \;&:=\; \langle\Omega|\mathsf{T}\big(\Phi(x_1)\,\Phi(x_2)\big)|\Omega\rangle - \langle\Omega|\mathsf{T}(\Phi(x_1))|\Omega\rangle\;\langle\Omega|\mathsf{T}(\Phi(x_2))|\Omega\rangle \\ &= \text{——} \;+\; \frac{1}{2}$$  $$+\; \frac{1}{2}$$  $$+\; \mathcal{O}(\lambda^3) \quad . \end{aligned} \tag{4.47}$$

It is worthwhile to note the similarity between the connected Feynman propagator and the concept of variance from probability theory. To translate this graphical description to an explicit formula, one can use the Feynman rules for Φ^3-theory:

- To every line between two points x and y assign a Feynman propagator

$$x \text{——} y \;=\; \Delta_F(x-y) \quad . \tag{4.48}$$

- To every vertex at a point z assign an integral

$$z \;=\; -\mathrm{i}\,\lambda \int_{\mathbb{R}^d} \mathrm{d}z \quad . \tag{4.49}$$

One then finds

$$\begin{aligned}\langle\Omega|\mathsf{T}\big(\Phi(x_1)\,\Phi(x_2)\big)|\Omega\rangle_{\mathrm{c}} \;&=\; \Delta_F(x_1-x_2) - \frac{\lambda^2}{2}\int_{\mathbb{R}^{2d}} \Delta_F(x_1-y)\,\Delta_F(x_2-z)\,\big(\Delta_F(y-z)\big)^2\,\mathrm{d}y\,\mathrm{d}z \\ &\quad - \frac{\lambda^2}{2}\int_{\mathbb{R}^{2d}} \Delta_F(x_1-y)\,\Delta_F(x_2-y)\,\Delta_F(y-z)\,\Delta_F(0)\,\mathrm{d}y\,\mathrm{d}z \;+\; \mathcal{O}(\lambda^3) \quad . \end{aligned} \tag{4.50}$$

Having a closer look at this expression, one recognizes as in Warning 4.5 the appearance of divergences, whose renormalization will be discussed later.

To conclude this section, let us study as in (4.43) the Fourier transform of this expression and in particular its pole structure. Performing a routine Fourier transform calculation, one finds that

$$\begin{aligned}\widetilde{\Delta}^{\mathrm{int}}_{F,c}(k) &= \widetilde{\Delta}_F(k) - \frac{\lambda^2}{2}\big(\widetilde{\Delta}_F(k)\big)^2 \int_{\mathbb{R}^d} \widetilde{\Delta}_F(q)\,\widetilde{\Delta}_F(k-q)\,\frac{\mathrm{d}q}{(2\pi)^d} \\ &\quad - \frac{\lambda^2}{2}\big(\widetilde{\Delta}_F(k)\big)^2\,\Delta_F(0)\,\widetilde{\Delta}_F(0) \;+\; \mathcal{O}(\lambda^3) \\ &= \widetilde{\Delta}_F(k)\,\Big(1 - \frac{\lambda^2}{2}\,\widetilde{\Delta}_F(k)\Big(\mathcal{I}(k^2)+\mathcal{J}\Big)\Big) \;+\; \mathcal{O}(\lambda^3) \quad , \end{aligned} \tag{4.51}$$

where in the last step have introduced the abbreviations $\mathcal{I}(k^2) := \int_{\mathbb{R}^d} \widetilde{\Delta}_F(q)\, \widetilde{\Delta}_F(k-q)\, \frac{\mathrm{d}q}{(2\pi)^d}$ and $\mathcal{J} := \Delta_F(0)\, \widetilde{\Delta}_F(0)$. Note that the integral $\mathcal{I}$ is Poincaré invariant, from which one can deduce that it has to be a function of k^2. Using the same geometric series trick as in (4.43), we can rewrite this as

$$\widetilde{\Delta}^{\mathrm{int}}_{F,c}(k) = \frac{-\mathrm{i}}{k^2 + m^2 - \mathrm{i}\,\frac{\lambda^2}{2}\,(\mathcal{I}(k^2) + \mathcal{J}) - \mathrm{i}\,\epsilon} \;+\; \mathcal{O}(\lambda^3) \quad . \tag{4.52}$$

The main difference between this result and (4.43) is that the correction term $\mathcal{I}(k^2)$ may depend on the Fourier momentum. This has the following consequence:

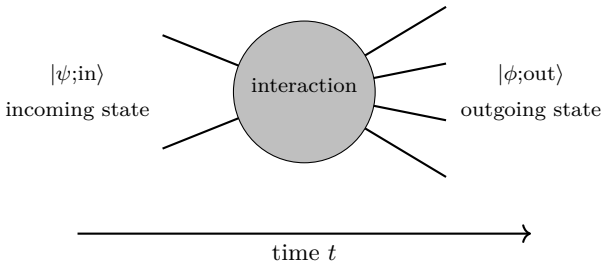
Observation 4.8. In analogy with Observation 4.7, the interacting Feynman propagator will have a pole at some value $k^2 = -m^2_{\mathrm{int}} \neq -m^2$ that is different from the mass parameter entering the classical action functional. Expanding the denominator in k^2 around this pole, one finds

$$Z^{-1} := 1 - \mathrm{i}\,\frac{\lambda^2}{2}\,\frac{\partial \mathcal{I}(k^2)}{\partial k^2}\bigg|_{k^2=-m^2_{\mathrm{int}}} \;+\; \mathcal{O}(\lambda^3) \quad , \tag{4.53}$$

which can be different from 1 since the correction term $\mathcal{I}(k^2)$ depends on k^2. The factor Z controls the value of the residue, in the sense that near the pole one has that $\widetilde{\Delta}^{\mathrm{int}}_{F,c}(k) \approx \frac{-\mathrm{i}\,Z}{k^2+m^2_{\mathrm{int}}-\mathrm{i}\,\epsilon}$. It is called, due to historical reasons, the *wave function renormalization* and we will see it again in the next section when discussing scattering theory.

4.4 Scattering and the LSZ formula

From a physical point of view, the most relevant observables that are associated with a QFT are *scattering amplitudes*. Loosely speaking, scattering means that one prepares in the far past $t \to -\infty$ a suitable state $|\psi;\mathrm{in}\rangle \in \mathcal{H}$ that describes a collection of incoming particles and asks the question what's the probability for these particles to interact with each other to produce in the far future $t \to +\infty$ another state $|\phi;\mathrm{out}\rangle \in \mathcal{H}$ that describes an in general different collection of outgoing particles. The following picture visualizes this scenario:



(4.54)

The probability for this scattering process to happen can be derived from the *scattering amplitude*

$$\langle \phi;\mathrm{out}|\psi;\mathrm{in}\rangle \;\in\; \mathbb{C} \tag{4.55}$$

that is defined in terms of the inner product of the Hilbert space. The aim of this section is to explain how such scattering amplitudes can be computed perturbatively in QFT.

The first aspect we have to clarify is how to define suitable asymptotic states $|\psi;\mathrm{in}\rangle \in \mathcal{H}$ and $|\phi;\mathrm{out}\rangle \in \mathcal{H}$ that admit an interpretation in terms of multiparticle states. Since the interacting field operator $\Phi(x)$ from (4.9) is *not* simply a Fourier integral over annihilation and creation operators (as it would be the case for a free field operator (4.4b)), it is not immediately obvious how to extract from it interacting multiparticle states. Assuming that all particles of interest are well separated in the far future/past such that interactions between them are negligible, it is reasonable to expect that the interacting field operator $\Phi(x)$ approaches a free field operator in the limit $t \to \pm\infty$. Heuristically, we may write this as

$$\Phi(x) \xrightarrow{t\to+\infty} \sqrt{Z}\, \Phi_{\mathrm{out}}(x) \quad , \qquad \Phi(x) \xrightarrow{t\to-\infty} \sqrt{Z}\, \Phi_{\mathrm{in}}(x) \quad , \tag{4.56}$$

where $\Phi_{\mathrm{out}}(x)$ and $\Phi_{\mathrm{in}}(x)$ are free quantum fields, but some mathematical care is needed to make precise the sense in which one takes limits of operators. The relevant concept here is called *weak limits* of operators, see e.g. Chapter 9.2 of the textbook by Greiner/Reinhardt for some details. The constant Z (called wave function renormalization) will in general be different from 1, which is due to the phenomenon from Observation 4.8 that self-interactions may modify the normalization of fields. Furthermore, the out and in quantum fields $\Phi_{\mathrm{out/in}}(x)$ will in general have a different mass $m^2_{\mathrm{int}} \neq m^2$ than the parameter m^2 from the classical action, which is due to the phenomenon from Observation 4.7 that self-interactions may modify the mass of fields. Since the out and in quantum fields are by hypothesis free, we can write them in terms of annihilation and creation operators

$$\Phi_{\mathrm{out/in}}(x) = \int_{\mathbb{R}^{d-1}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathrm{out/in}}(\mathbf{k})\, e^{\mathrm{i}\, k\, x} + a^\dagger_{\mathrm{out/in}}(\mathbf{k})\, e^{-\mathrm{i}\, k\, x} \right) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \tag{4.57}$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2_{\mathrm{int}}}$ involves the interacting mass. We can now use these out/in annihilation and creation operators to define in analogy with (3.41) a state

$$|k_1, \ldots, k_n; \mathrm{in}\rangle \;:=\; \sqrt{2\omega_{\mathbf{k}_1}} \cdots \sqrt{2\omega_{\mathbf{k}_n}}\, a^\dagger_{\mathrm{in}}(\mathbf{k}_1) \cdots a^\dagger_{\mathrm{in}}(\mathbf{k}_n)|\Omega\rangle \tag{4.58}$$

that describes n incoming particles with relativistic momenta k_a satisfying the on-shell condition $k_a^2 = -m^2_{\mathrm{int}}$ and a state

$$|q_1, \ldots, q_m; \mathrm{out}\rangle \;:=\; \sqrt{2\omega_{\mathbf{q}_1}} \cdots \sqrt{2\omega_{\mathbf{q}_m}}\, a^\dagger_{\mathrm{out}}(\mathbf{q}_1) \cdots a^\dagger_{\mathrm{out}}(\mathbf{q}_m)|\Omega\rangle \tag{4.59}$$

that describes m outgoing particles with relativistic momenta q_b satisfying the on-shell condition $q_b^2 = -m^2_{\mathrm{int}}$. (Here $|\Omega\rangle$ denotes the interacting vacuum state from Section 4.1, which we recall is in general different from the free vacuum state $|0\rangle$.)

The goal for the rest of this section is to express the scattering amplitude

$$\langle q_1, \ldots, q_m; \mathrm{out}|k_1, \ldots, k_n; \mathrm{in}\rangle \tag{4.60}$$

in terms of the time-ordered $n+m$-point function of the interacting quantum field $\Phi(x)$. To simplify our arguments, we make the technical assumption that the outgoing relativistic momenta are all different from the incoming ones, i.e. $q_b \neq k_a$ for all $b = 1, \ldots, m$ and $a = 1, \ldots, n$. Physically speaking, this means that we exclude the uninteresting case where one or more of the incoming particles do not participate in the interaction and simply act as spectators for the interaction among the other particles. (Physicists would say that we exclude forward scattering.) To achieve our goal,

we require some preparations. First, we note that the out/in annihilation and creation operators can be determined by the following integrals

$$\sqrt{2\omega_{\mathbf{k}}}\, a_{\text{out/in}}(\mathbf{k}) \;=\; \mathrm{i}\int_{\mathbb{R}^{d-1}} e^{-\mathrm{i}\,k\,x} \overleftrightarrow{\partial_0}\, \Phi_{\text{out/in}}(x)\; \mathrm{d}\mathbf{x} \quad , \tag{4.61a}$$

$$\sqrt{2\omega_{\mathbf{k}}}\, a^\dagger_{\text{out/in}}(\mathbf{k}) \;=\; -\mathrm{i}\int_{\mathbb{R}^{d-1}} e^{\mathrm{i}\,k\,x} \overleftrightarrow{\partial_0}\, \Phi_{\text{out/in}}(x)\; \mathrm{d}\mathbf{x} \quad , \tag{4.61b}$$

where the right-left derivative operator $\overleftrightarrow{\partial_0}$ is defined, for any two functions $f(x)$ and $g(x)$ on the Minkowski spacetime $x \in \mathbb{R}^d$, by

$$f(x) \overleftrightarrow{\partial_0} g(x) \;:=\; f(x)\,\frac{\partial g(x)}{\partial t} - \frac{\partial f(x)}{\partial t}\, g(x) \quad . \tag{4.62}$$

Combining this with our hypothesis in (4.56), we can extract the out/in annihilation and creation operators directly from the interacting field operator $\Phi(x)$ by taking suitable limits

$$\sqrt{2\omega_{\mathbf{k}}}\, a_{\text{out/in}}(\mathbf{k}) \;=\; \frac{\mathrm{i}}{\sqrt{Z}} \lim_{t\to\pm\infty} \int_{\mathbb{R}^{d-1}} e^{-\mathrm{i}\,k\,x} \overleftrightarrow{\partial_0}\, \Phi(x)\; \mathrm{d}\mathbf{x} \quad , \tag{4.63a}$$

$$\sqrt{2\omega_{\mathbf{k}}}\, a^\dagger_{\text{out/in}}(\mathbf{k}) \;=\; -\frac{\mathrm{i}}{\sqrt{Z}} \lim_{t\to\pm\infty} \int_{\mathbb{R}^{d-1}} e^{\mathrm{i}\,k\,x} \overleftrightarrow{\partial_0}\, \Phi(x)\; \mathrm{d}\mathbf{x} \quad . \tag{4.63b}$$

For later use, we observe that the difference between the out and in annihilation operator can be written in a particularly useful form

$$\begin{aligned}
\sqrt{2\omega_{\mathbf{k}}}\,\Big(a_{\text{out}}(\mathbf{k}) - a_{\text{in}}(\mathbf{k})\Big) \;&=\; \frac{\mathrm{i}}{\sqrt{Z}}\Big(\lim_{t\to+\infty} - \lim_{t\to-\infty}\Big) \int_{\mathbb{R}^{d-1}} e^{-\mathrm{i}\,k\,x} \overleftrightarrow{\partial_0}\, \Phi(x)\; \mathrm{d}\mathbf{x} \\
&=\; \frac{\mathrm{i}}{\sqrt{Z}} \int_{\mathbb{R}^d} \partial_0\Big(e^{-\mathrm{i}\,k\,x} \overleftrightarrow{\partial_0}\, \Phi(x)\Big)\; \mathrm{d}x \\
&=\; \frac{\mathrm{i}}{\sqrt{Z}} \int_{\mathbb{R}^d} \Big(e^{-\mathrm{i}\,k\,x}\,\partial_0^2\Phi(x) - \partial_0^2\Big(e^{-\mathrm{i}\,k\,x}\Big)\,\Phi(x)\Big)\; \mathrm{d}x \\
&=\; \frac{\mathrm{i}}{\sqrt{Z}} \int_{\mathbb{R}^d} \Big(e^{-\mathrm{i}\,k\,x}\,\partial_0^2\Phi(x) - (\nabla^2 - m^2_{\text{int}})\Big(e^{-\mathrm{i}\,k\,x}\Big)\,\Phi(x)\Big)\; \mathrm{d}x \\
&=\; \frac{\mathrm{i}}{\sqrt{Z}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\,k\,x}\,(-\partial^2 + m^2_{\text{int}})\Phi(x)\; \mathrm{d}x \quad .
\end{aligned} \tag{4.64}$$

In the second line we have used that $\int_{-\infty}^{\infty} \partial_0(\cdots)\,\mathrm{d}t = \lim_{t\to+\infty}(\cdots) - \lim_{t\to-\infty}(\cdots)$. The fourth line makes use of the Klein-Gordon equation $(-\partial^2 + m^2_{\text{int}})e^{-\mathrm{i}\,k\,x} = 0$ and the last line follows via integration by parts. Taking the adjoint of this expression, we obtain a similar formula

$$\sqrt{2\omega_{\mathbf{k}}}\,\Big(a^\dagger_{\text{out}}(\mathbf{k}) - a^\dagger_{\text{in}}(\mathbf{k})\Big) = -\frac{\mathrm{i}}{\sqrt{Z}} \int_{\mathbb{R}^d} e^{\mathrm{i}\,k\,x}\,(-\partial^2 + m^2_{\text{int}})\Phi(x)\; \mathrm{d}x \tag{4.65}$$

for the creation operators.

With these preparations, we can now rewrite the scattering amplitude (4.60) in a more useful

way. This is an iterative procedure, starting with the observation that we can use (4.65) to write

$$\begin{aligned}\langle q_1,\dots,q_m;\text{out}|k_1,\dots,k_n;\text{in}\rangle &= \sqrt{2\omega_{\mathbf{k}_1}}\,\langle q_1,\dots,q_m;\text{out}|a^\dagger_{\text{in}}(\mathbf{k}_1)|k_2,\dots,k_n;\text{in}\rangle \\ &= \frac{\mathrm{i}}{\sqrt{Z}}\int_{\mathbb{R}^d} e^{\mathrm{i}\,k_1\,x_1}\,(-\partial^2_{x_1}+m^2_{\text{int}})\langle q_1,\dots,q_m;\text{out}|\Phi(x_1)|k_2,\dots,k_n;\text{in}\rangle\,\mathrm{d}x_1 \\ &\quad + \sqrt{2\omega_{\mathbf{k}_1}}\,\langle q_1,\dots,q_m;\text{out}|a^\dagger_{\text{out}}(\mathbf{k}_1)|k_2,\dots,k_n;\text{in}\rangle \\ &= \frac{\mathrm{i}}{\sqrt{Z}}\int_{\mathbb{R}^d} e^{\mathrm{i}\,k_1\,x_1}\,(-\partial^2_{x_1}+m^2_{\text{int}})\langle q_1,\dots,q_m;\text{out}|\Phi(x_1)|k_2,\dots,k_n;\text{in}\rangle\,\mathrm{d}x_1 \quad , \end{aligned} \tag{4.66}$$

where in the last step we have used our hypothesis that $k_1 \neq q_b$ for all $b = 1,\dots,m$, which means that we can pull $a^\dagger_{\text{out}}(\mathbf{k}_1)$ through all the annihilation operators $a_{\text{out}}(\mathbf{q}_b)$ on its left to annihilate the vacuum $\langle\Omega|a^\dagger_{\text{out}}(\mathbf{k}_1) = \langle a_{\text{out}}(\mathbf{k}_1)\Omega| = 0$. To perform the next iterative steps, let us note that

$$\begin{aligned}\sqrt{2\omega_{\mathbf{k}_2}}\,\Phi(x_1)\,a^\dagger_{\text{in}}(\mathbf{k}_2) &= \sqrt{2\omega_{\mathbf{k}_2}}\,\mathsf{T}\Big(\Phi(x_1)\,a^\dagger_{\text{in}}(\mathbf{k}_2)\Big) \\ &= \frac{\mathrm{i}}{\sqrt{Z}}\int_{\mathbb{R}^d} e^{\mathrm{i}\,k_2\,x_2}\,(-\partial^2_{x_2}+m^2_{\text{int}})\mathsf{T}\Big(\Phi(x_1)\,\Phi(x_2)\Big)\,\mathrm{d}x_2 \\ &\quad + \sqrt{2\omega_{\mathbf{k}_2}}\,\mathsf{T}\Big(\Phi(x_1)\,a^\dagger_{\text{out}}(\mathbf{k}_2)\Big) \\ &= \frac{\mathrm{i}}{\sqrt{Z}}\int_{\mathbb{R}^d} e^{\mathrm{i}\,k_2\,x_2}\,(-\partial^2_{x_2}+m^2_{\text{int}})\mathsf{T}\Big(\Phi(x_1)\,\Phi(x_2)\Big)\,\mathrm{d}x_2 \\ &\quad + \sqrt{2\omega_{\mathbf{k}_2}}\,a^\dagger_{\text{out}}(\mathbf{k}_2)\,\Phi(x_1) \quad . \end{aligned} \tag{4.67}$$

The last term will again not contribute to the scattering amplitude because $a^\dagger_{\text{out}}(\mathbf{k}_2)$ can be pulled through all the annihilation operators $a_{\text{out}}(\mathbf{q}_b)$ on its left to annihilate the vacuum. By a similar computation (using (4.64) instead of (4.65)), one can show that also the out annihilation operators arising from $\langle q_1,\dots,q_m;\text{out}| = \langle q_2,\dots,q_m;\text{out}|\,\sqrt{2\omega_{\mathbf{q}_1}}\,a_{\text{out}}(\mathbf{q}_1)$ add interacting field operators to the time-ordered product, up to in annihilation operators that annihilate the vacuum to their right. (It is a good exercise for you to check this!) Hence, we can iteratively take care of all k_a and all q_b in the scattering amplitude and obtain

$$\begin{aligned}&\langle q_1,\dots,q_m;\text{out}|k_1,\dots,k_n;\text{in}\rangle = \Big(\frac{\mathrm{i}}{\sqrt{Z}}\Big)^{n+m}\int_{\mathbb{R}^{(n+m)d}} e^{\mathrm{i}\sum_{a=1}^n k_a\,x_a - \mathrm{i}\sum_{b=1}^m q_b\,y_b}\,\times \\ &\prod_{a=1}^n(-\partial^2_{x_a}+m^2_{\text{int}})\prod_{b=1}^m(-\partial^2_{y_b}+m^2_{\text{int}})\,\langle\Omega|\mathsf{T}\Big(\prod_{a=1}^n\Phi(x_a)\prod_{b=1}^m\Phi(y_b)\Big)|\Omega\rangle\,\mathrm{d}x_1\cdots\mathrm{d}x_n\,\mathrm{d}y_1\cdots\mathrm{d}y_m \quad , \end{aligned} \tag{4.68}$$

which is called the *LSZ reduction formula*, named after Lehmann, Symanzik and Zimmermann. This formula is indeed very useful because it reduces the description of scattering amplitudes to the interacting time-ordered $n+m$-point functions, which we already know how to determine thanks to the Gell-Mann and Low reduction formula (4.16) and Wick's theorem (4.32).

It is often more convenient to write the LSZ reduction formula in Fourier space. Denoting the Fourier transform of the time-ordered n-point function by

$$\widetilde{G}_n(k_1,\dots,k_n) := \int_{\mathbb{R}^{nd}} e^{-\mathrm{i}\sum_{a=1}^n k_a\,x_a}\,\langle\Omega|\mathsf{T}\Big(\Phi(x_1)\cdots\Phi(x_n)\Big)|\Omega\rangle\,\mathrm{d}x_1\cdots\mathrm{d}x_n \quad , \tag{4.69}$$

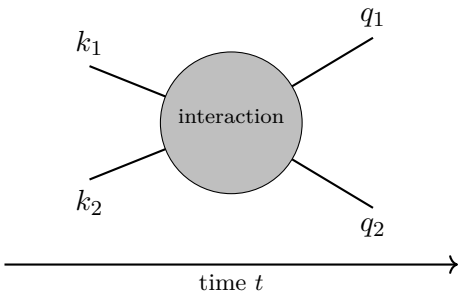
one finds that

$$\langle q_1,\dots,q_m;\text{out}|k_1,\dots,k_n;\text{in}\rangle = \Big(\prod_{a=1}^{n}\frac{k_a^2+m_{\text{int}}^2}{-\text{i}\sqrt{Z}}\Big)\Big(\prod_{b=1}^{m}\frac{q_b^2+m_{\text{int}}^2}{-\text{i}\sqrt{Z}}\Big)\,\widetilde{G}_{n+m}(-k_1,\dots,-k_n,q_1,\dots,q_m) \quad . \tag{4.70}$$

Warning 4.9. Recall that the relativistic momenta k_a and q_b of the incoming and the going particles satisfy the on-shell conditions $k_a^2=-m_{\text{int}}^2$ and $q_b^2=-m_{\text{int}}^2$. But wait! Doesn't this imply that all factors of the products in (4.70) are zero and hence the whole scattering amplitude is zero? This is luckily not the case, because the Fourier transform $\widetilde{G}_{n+m}(-k_1,\dots,-k_n,q_1,\dots,q_m)$ of the time-ordered $n+m$-point function can have poles that cancel these zeros! In the next section we will see in detail how this cancellation works in examples, but the basic mechanism is already understandable at this point by recalling that the interacting Feynman propagator in Fourier space has a pole at $k^2=-m_{\text{int}}^2$, see Observation 4.7. Summing up, this means that evaluating the LSZ formula in Fourier space (4.70) requires some care: One computes the products on the right-hand side of the equation (4.70) for general Fourier momenta k_a and q_b, which do not necessarily satisfy the on-shell condition, and only at the very end studies the limit of the total expression when going on-shell $k_a^2=-m_{\text{int}}^2$ and $q_b^2=-m_{\text{int}}^2$.

4.5 Simple examples of scattering amplitudes

To conclude this chapter, we shall work out some examples of scattering amplitudes to the lowest order in the coupling constants. We shall focus on $2\to 2$ scatting amplitudes $\langle q_1,q_2;\text{out}|k_1,k_2;\text{in}\rangle$, which means that there are two incoming particles with relativistic momenta $k_1,k_2\in\mathbb{R}^d$ that interact with each other and produce two outgoing particles with different relativistic momenta $q_1,q_2\in\mathbb{R}^d$. In pictures, this means that we consider scatterings of the following type



(4.71)

where, as we shall see below, the grey blob admits a description in terms of Feynman diagrams.

Since our current goal is to describe scattering amplitudes to the lowest nontrivial order in the coupling constants, we can set in the LSZ formula (4.68), and in its Fourier space version (4.70), the wave function renormalization $Z=1$ to be one and the interacting mass $m_{\text{int}}^2=m^2$ to be the parameter from the classical action functional. This is consistent because the corrections to Z and m_{int}^2 are of higher order in the coupling constants, see Observations 4.7 and 4.8. (It is however important to emphasize that such corrections to Z and m_{int}^2 are important for computing

consistently the higher order corrections to scattering amplitudes.) This reduces the problem to computing the time-ordered 4-point function

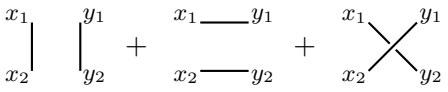
$$\langle\Omega|\mathsf{T}(\Phi(x_1)\Phi(x_2)\Phi(y_1)\Phi(y_2))|\Omega\rangle \tag{4.72}$$

or, in view of the LSZ formula in Fourier space (4.70), its Fourier transform

$$\begin{aligned}\widetilde{G}_4(-k_1,-k_2,q_1,q_2) = \int_{\mathbb{R}^{4d}} e^{\mathrm{i}\,(k_1x_1+k_2x_2-q_1y_1-q_2y_2)}\ \times \\ \langle\Omega|\mathsf{T}\Big(\Phi(x_1)\Phi(x_2)\Phi(y_1)\Phi(y_2)\Big)|\Omega\rangle\ \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}y_2 \quad .\end{aligned} \tag{4.73}$$

Let us start by analyzing the problem at order λ^0 in the coupling constants (i.e. for a free QFT) in order to get some feeling for the zero vs. pole issue from Warning 4.9. In this case the interacting time-ordered 4-point function coincides with the free one. Using also Example 4.4, we obtain

$$\begin{aligned}\langle\Omega|\mathsf{T}\Big(\Phi(x_1)\Phi(x_2)\Phi(y_1)\Phi(y_2)\Big)|\Omega\rangle &= \langle 0|\mathsf{T}\Big(\Phi_0(x_1)\Phi_0(x_2)\Phi_0(y_1)\Phi_0(y_2)\Big)|0\rangle + \mathcal{O}(\lambda^1) \\ &= (\Delta_F)_{x_1x_2}\,(\Delta_F)_{y_1y_2} + (\Delta_F)_{x_1y_1}\,(\Delta_F)_{x_2y_2} + (\Delta_F)_{x_1y_2}\,(\Delta_F)_{x_2y_1} + \mathcal{O}(\lambda^1) \\ &= \end{aligned}$$



$$+\ \mathcal{O}(\lambda^1) \quad . \tag{4.74}$$

It is important to note that there is no interaction vertex in the third diagram; the two crossing lines are a graphical representation of the product of Feynman propagators $(\Delta_F)_{x_1y_2}\,(\Delta_F)_{x_2y_1}$. Passing over to Fourier space, we obtain

$$\begin{aligned}\widetilde{G}_4(-k_1,-k_2,q_1,q_2) = (2\pi)^{2d}\Bigg(&\frac{-\mathrm{i}}{k_1^2+m^2-\mathrm{i}\,\epsilon}\,\frac{-\mathrm{i}}{q_1^2+m^2-\mathrm{i}\,\epsilon}\,\delta(k_1+k_2)\,\delta(q_1+q_2) \\ &+ \frac{-\mathrm{i}}{k_1^2+m^2-\mathrm{i}\,\epsilon}\,\frac{-\mathrm{i}}{k_2^2+m^2-\mathrm{i}\,\epsilon}\,\delta(k_1-q_1)\,\delta(k_2-q_2) \\ &+ \frac{-\mathrm{i}}{k_1^2+m^2-\mathrm{i}\,\epsilon}\,\frac{-\mathrm{i}}{k_2^2+m^2-\mathrm{i}\,\epsilon}\,\delta(k_1-q_2)\,\delta(k_2-q_1)\Bigg) + \mathcal{O}(\lambda^1) \quad ,\end{aligned} \tag{4.75}$$

where we note that the Dirac delta functions enforce the conservation of the relativistic momentum along the lines drawn in our Feynman diagrams. Inserting this into the LSZ formula in Fourier space (4.70) however gives zero, because the two poles of $\widetilde{G}_4(-k_1,-k_2,q_1,q_2)$ are not sufficient to compensate the four zeros in its prefactor. Hence, we find that the scattering amplitude

$$\langle q_1,q_2;\mathrm{out}|k_1,k_2;\mathrm{in}\rangle = 0 + \mathcal{O}(\lambda^1) \tag{4.76}$$

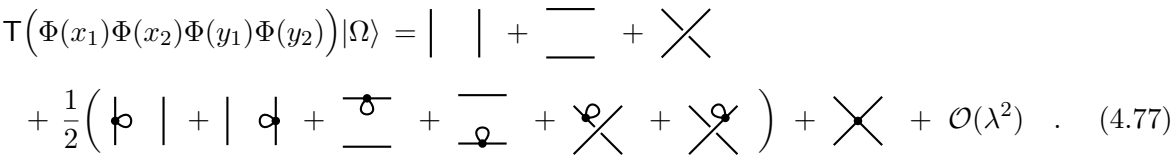
vanishes in the case of no interactions, which is of course totally sensible since scattering should happen only when the particles interact with each other.

In order to get some nontrivial scattering amplitudes, we shall again study two examples of interaction terms, namely Φ^4-theory and Φ^3-theory as in Section 4.3.

Φ^4-theory:

Using Wick's theorem (4.32) and the Observation 4.6 that vacuum bubbles do not contribute to the computation of the interacting time-ordered n-point functions, one finds that the interacting time-ordered 4-point function of Φ^4-theory to first order in λ is given by

$\langle\Omega|\mathsf{T}\Big(\Phi(x_1)\Phi(x_2)\Phi(y_1)\Phi(y_2)\Big)|\Omega\rangle$ =



$+ \mathcal{O}(\lambda^2)$. (4.77)

Passing over to Fourier space and inserting the result into the LSZ formula (4.70), we observe that only the last diagram contributes to the scattering amplitude because all other terms do not have sufficiently many poles to compensate the four zeros in the prefactor of (4.70). Performing a routine Fourier transform calculation, one finds

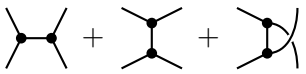
$$\langle q_1, q_2; \text{out}|k_1, k_2; \text{in}\rangle = -\mathrm{i}\,\lambda\,(2\pi)^d\,\delta(k_1+k_2-q_1-q_2) \;+\; \mathcal{O}(\lambda^2) \quad . \tag{4.78}$$

Note again that there is a Dirac delta function that enforces relativistic momentum conservation, i.e. the total incoming momentum $k_1 + k_2$ must be the same as the total outgoing momentum $q_1 + q_2$ for the scattering to take place. Furthermore, the coupling constant λ, which controls the interaction term in the action functional $S_{\text{int}}[\Phi] = -\frac{\lambda}{4!}\int_{\mathbb{R}^d}\Phi^4(x)\,\mathrm{d}x$, determines the magnitude of the scattering amplitude.

Φ^3-theory:

Using again Wick's theorem (4.32) and the Observation 4.6 that vacuum bubbles do not contribute to the computation of the interacting time-ordered n-point functions, one finds that the interacting time-ordered 4-point function of Φ^3-theory to second order in λ is given by

$\langle\Omega|\mathsf{T}\Big(\Phi(x_1)\Phi(x_2)\Phi(y_1)\Phi(y_2)\Big)|\Omega\rangle$ =



$+ \mathcal{O}(\lambda^3)$

$+$ more diagrams that do not contribute to the scattering amplitude (4.70) . (4.79)

It is a good exercise for you to determine these additional diagrams and try to understand why they do not contribute to the scattering amplitude (4.70). Performing once more a routine Fourier transform calculation, one finds

$$\begin{aligned}\langle q_1, q_2; \text{out}|k_1, k_2; \text{in}\rangle &= (-\mathrm{i}\,\lambda)^2\,(2\pi)^d\,\delta(k_1+k_2-q_1-q_2)\;\times \\ &\left(\frac{-\mathrm{i}}{(k_1+k_2)^2+m^2}+\frac{-\mathrm{i}}{(k_1-q_1)^2+m^2}+\frac{-\mathrm{i}}{(k_1-q_2)^2+m^2}\right) \;+\; \mathcal{O}(\lambda^3) \quad .\end{aligned} \tag{4.80}$$

Note that the *internal lines* of Feynman diagrams contribute with a Feynman propagator (in Fourier space), whose momentum is determined by relativistic momentum conservation at each interaction vertex. These internal lines are interpreted as *virtual particles*, because their relativistic momenta do not necessarily satisfy the on-shell conditions, i.e. $(k_1+k_2)^2 \neq -m^2$, $(k_1-q_1)^2 \neq -m^2$ and $(k_1-q_2)^2 \neq -m^2$. Such virtual particles are a quantum mechanical phenomenon that does not have any analog in classical physics. The mechanism that interactions between particles are mediated by virtual particles is very common in QFT and it also appears in physically relevant examples, such as quantum electrodynamics (QED) or the standard model of particle physics.

4.6 Feynman rules for real scalar QFTs

Our calculations in the previous sections may seem a bit cumbersome at first sight, but they all follow the same pattern which can be summarized by a small number of *Feynman rules*. Using such Feynman rules, one can avoid redoing again and again the same type of calculations that are needed to determine from the Gell-Mann and Low formula (4.16) and Wick's theorem (4.32) the interacting time-ordered n-point functions and the scattering amplitudes of a QFT. To generalize a bit what was written in the previous sections, let us consider here the case of Φ^N-theory for an arbitrary $N \geq 2$, i.e. we work with the interaction term

$$S_{\text{int}}[\Phi_0] = -\frac{\lambda}{N!} \int_{\mathbb{R}^d} \big(\Phi_0(x)\big)^N \, \mathrm{d}x \quad . \tag{4.81}$$

Time-ordered n-point functions:

The interacting time-ordered n-point function

$$\langle \Omega | \mathsf{T}\big(\Phi(x_1) \cdots \Phi(x_n)\big) | \Omega \rangle \tag{4.82}$$

at order λ^k can be computed by carrying out the following steps:

- Draw n external points $x_1, \ldots, x_n$ and k internal points $y_1, y_2, \ldots, y_k$. The latter represent the k interaction vertices arising at order λ^k.
- Connect the points by lines such that
  - 1 line is attached to each external point x_i,
  - N lines are attached to each internal point y_j.

  In general, there will be different ways in which one can connect the points, leading to different Feynman diagrams.
- Record all (topologically distinct) Feynman diagrams and discard those including vacuum bubbles, see Observation 4.6.
- Associate to each Feynman diagram an analytical expression by assigning
  - to each external point x_i a factor 1,
  - to each internal point y_j a factor $-\mathrm{i}\,\lambda \int_{\mathbb{R}^d} \mathrm{d}y_j$,
  - to each line between two points z_a and z_b (these can be external or internal) a free Feynman propagator $\Delta_F(z_a - z_b)$.

  These factors and Feynman propagators are then multiplied and weighted by the symmetry factor of the diagram. Note that the ordering of points in the Feynman propagators does not matter for a real scalar field because $\Delta_F(z_a - z_b) = \Delta_F(z_b - z_a)$.
- Sum up the individual contributions associated with distinct Feynman diagrams.

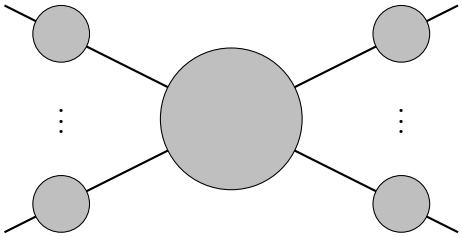
It would be a good exercise for you to redo our calculations in Section 4.3 by using directly the Feynman rules.

Scattering amplitudes:

Computing $n \to m$ scattering amplitudes

$$\langle q_1, \dots, q_m; \text{out} | k_1, \dots, k_n; \text{in} \rangle \tag{4.83}$$

is related to computing time-ordered $n+m$-point functions, but there are some slight differences due to the prefactors in the LSZ formula (4.70) which have the effect of "amputating external legs". As we have already seen in Section 4.5, only those Feynman diagrams in which each external point is connected to an internal point survive this amputation since the zeros in the prefactors of the LSZ formula turn all other contributions to zero when going on-shell. When considering higher orders in the coupling constant, it is also important to note that the amputation described by the LSZ formula (4.70) involves the interacting mass m^2_{int} and (the square root of) the wave function renormalization Z, hence it takes care of the loop corrections the particles receive while propagating to or from the interaction region. This reduces further the number of Feynman diagrams that contribute to the scattering amplitude. To understand which Feynman diagrams we should take into account, let us note that any Feynman diagram can be factorized as

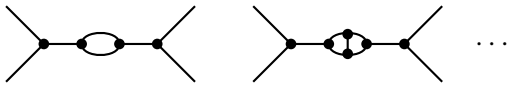


(4.84)

The big blob in the middle describes the actual particle interaction and the small blobs on the external legs describe the loop corrections received by the particles while propagating to or from the interaction region. Since the latter are the source of the interacting mass m^2_{int} and the wave function renormalization Z, they will be amputated by the prefactors in the LSZ formula (4.70), leading to

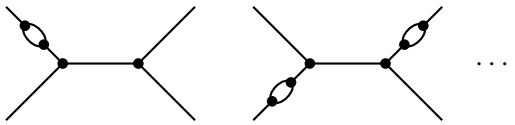
$$\langle q_1, \dots, q_m; \text{out} | k_1, \dots, k_n; \text{in} \rangle \;=\; (\sqrt{Z})^{n+m} \;\; \widetilde{G}_{n+m}(-k_1, \dots, -k_n, q_1, \dots, q_m)\Big|_{\text{amputated}} \quad . \tag{4.85}$$

Hence, only those Feynman diagrams with a trivial blob on each external leg have to be considered for computing scattering amplitudes. For example, for $2 \to 2$ scattering in Φ^3-theory, diagrams such as



(4.86)

have to be considered, while diagrams such as



(4.87)

do not have to be considered. It is also important to note the prefactor $\sqrt{Z}^{n+m}$ in the amputated LSZ formula (4.85), which arises because the residue of each interacting Feynman propagator is

by definition Z (see Observation 4.8) while the LSZ formula (4.70) only divides by $1/\sqrt{Z}$ for each external leg. As a consequence, one has to know the corrections to $Z = 1 + \mathcal{O}(\lambda)$, which are determined by computing the interacting Feynman propagator, in order to compute higher order corrections to scattering amplitudes. Luckily, if one is only interested in scattering amplitudes at leading order in the coupling constant λ, as we were in our examples in Section 4.5, one can safely set $Z = 1$.

With these preparations, we can now write down the Feynman rules for scattering amplitudes, which we will do immediately in Fourier space since this is practically more convenient. The $n \to m$ scattering amplitude (4.83) with k interaction vertices can be computed by carrying out the following steps:

- Draw n external points on the left, m external points on the right and k internal points in the middle. These points represent, respectively, the incoming particles, the outgoing particles and the interaction vertices.

- In analogy to the case of time-ordered correlation functions, connect these points by lines to produce Feynman diagrams (excluding vacuum bubbles). Due to the amputation of external legs, consider only those Feynman diagrams in which 1.) each external point is connected by a line to an internal point and 2.) the blob on each external leg is trivial.

- Decorate the lines of these Feynman diagrams with relativistic momenta by assigning
  - the incoming momenta k_a (pointing towards the interaction region) to the incoming external legs,
  - the outgoing momenta q_b (pointing away from the interaction region) to the outgoing external legs,
  - an arbitrary momentum l_c (pointing in the direction you like) to each internal line c.

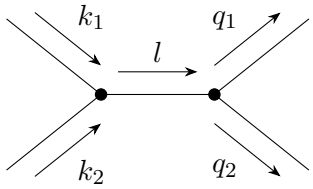
- Associate to each decorated Feynman diagram an analytical expression by assigning
  - to each (incoming or outgoing) external leg a factor of $\sqrt{Z}$,
  - to each internal line with momentum l_c a free Feynman propagator $\frac{-\mathrm{i}}{l_c^2+m^2-\mathrm{i}\,\epsilon}$,
  - to each interaction vertex a factor $-\mathrm{i}\,\lambda\,(2\pi)^d\,\delta(p_{\text{in}} - p_{\text{out}})$ with p_{in} the total incoming momentum at the vertex and p_{out} the total outgoing momentum at the vertex. (This enforces relativistic momentum conservation at each vertex.)

  These factors and Feynman propagators are then multiplied, weighted by the symmetry factor of the diagram and integrated over all internal momenta $\prod_c \int_{\mathbb{R}^d} \frac{\mathrm{d}l_c}{(2\pi)^d}$.

- Sum up the individual contributions associated with distinct Feynman diagrams.

You might be surprised by the "momentum decoration rules" for Feynman diagrams, which is something we haven't explicitly used in our (more rudimentary) diagrammatics in Section 4.5. The reason for this is that all diagrams in Section 4.5 are so simple that it is immediately obvious how to assign a momentum to each line by enforcing relativistic momentum conservation at all vertices. This won't be the case anymore for more complicated diagrams, which is why the "momentum decoration rules" are a useful concept.

To practice the Feynman rules for scattering amplitudes, let us redo in detail the first diagram in (4.79). (This diagram arises for $2 \to 2$ scattering in Φ^3-theory.) We redraw this diagram with momentum decoration



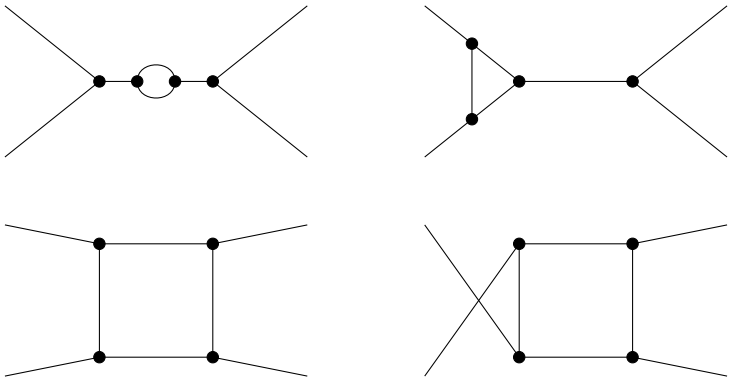
(4.88a)

which when evaluated with the Feynman rules gives

$$\begin{aligned} Z^2 \int_{\mathbb{R}^d} (-\mathrm{i}\,\lambda)\,(2\pi)^d\,\delta(k_1+k_2-l)\;\frac{-\mathrm{i}}{l^2+m^2-\mathrm{i}\,\epsilon}\;(-\mathrm{i}\,\lambda)\,(2\pi)^d\,\delta(l-q_1-q_2)\;\frac{\mathrm{d}l}{(2\pi)^d} \\ = Z^2\,(-\mathrm{i}\,\lambda)^2\,(2\pi)^d\,\delta(k_1+k_2-q_1-q_2)\;\frac{-\mathrm{i}}{(k_1+k_2)^2+m^2-\mathrm{i}\,\epsilon} \quad . \end{aligned} \tag{4.88b}$$

Using that $Z = 1 + \mathcal{O}(\lambda)$ and sending $\epsilon \to 0$, we find the same result as previously in the first term of (4.80). The other diagrams in (4.79) can be treated similarly, which allows us to recover the result (4.80) by using Feynman rules.

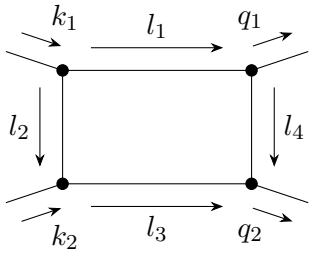
Going to higher orders in the coupling constant, there are of course many more Feynman diagrams one has to consider. It is a nontrivial task to write all of them down in a systematic fashion, which is why QFT practitioners are typically using computer assistance to ensure that they do not forget diagrams. To see some examples of such higher order Feynman diagrams for scattering amplitudes, let us consider again $2 \to 2$ scattering in Φ^3-theory. Among the many diagrams that contribute at order λ^4, we have the following examples



(4.89)

A common feature of all these diagrams is that they contain closed loops, which in many cases will cause divergences as previously seen in Warning 4.5. These divergences will be treated later in Chapter 8 when we introduce renormalization techniques. To practice once more the application of Feynman rules, let us evaluate the "box diagram" in the bottom left corner: First of all, we add

relativistic momentum decorations to this diagram



(4.90)

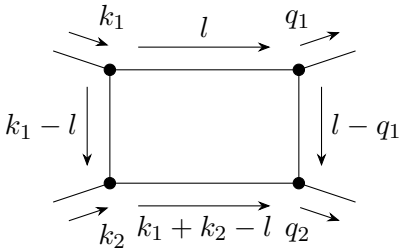
Applying the Feynman rules then gives the expression

$$\sqrt{Z}^4 \int_{\mathbb{R}^{4d}} \frac{-i}{l_1^2+m^2-\mathrm{i}\,\epsilon}\,\frac{-i}{l_2^2+m^2-\mathrm{i}\,\epsilon}\,\frac{-i}{l_3^2+m^2-\mathrm{i}\,\epsilon}\,\frac{-i}{l_4^2+m^2-\mathrm{i}\,\epsilon}\,\times$$
$$(-\mathrm{i}\,\lambda)\,(2\pi)^d\,\delta(k_1-l_1-l_2)\;(-\mathrm{i}\,\lambda)\,(2\pi)^d\,\delta(k_2+l_2-l_3)\,\times$$
$$(-\mathrm{i}\,\lambda)\,(2\pi)^d\,\delta(l_3+l_4-q_2)\;(-\mathrm{i}\,\lambda)\,(2\pi)^d\,\delta(l_1-q_1-l_4)\,\times$$
$$\frac{\mathrm{d}l_1}{(2\pi)^d}\,\frac{\mathrm{d}l_2}{(2\pi)^d}\,\frac{\mathrm{d}l_3}{(2\pi)^d}\,\frac{\mathrm{d}l_4}{(2\pi)^d}\quad . \tag{4.91}$$

Carrying out the momentum space integrals over l_2, l_3, l_4 by using that "delta functions kill integrations", one obtains the following simplified expression

$$Z^2\,(-\mathrm{i}\,\lambda)^4\,(2\pi)^d\,\delta(k_1+k_2-q_1-q_2)\int_{\mathbb{R}^d}\frac{-\mathrm{i}}{l^2+m^2-\mathrm{i}\,\epsilon}\,\frac{-\mathrm{i}}{(l-q_1)^2+m^2-\mathrm{i}\,\epsilon}\,\times$$
$$\frac{-\mathrm{i}}{(k_1-l)^2+m^2-\mathrm{i}\,\epsilon}\,\frac{-\mathrm{i}}{(k_1+k_2-l)^2+m^2-\mathrm{i}\,\epsilon}\,\frac{\mathrm{d}l}{(2\pi)^d}\quad . \tag{4.92}$$

There is again a Dirac delta function that enforces overall relativistic momentum conservation for the external legs ($k_1+k_2=q_1+q_2$), which is a common feature of all scattering amplitudes that is linked to the Poincaré invariance of our QFTs. The momenta entering the four Feynman propagators in (4.92) have been determined by using relativistic momentum conservation at each interaction vertex, which is enforced by the delta functions associated to vertices in (4.91). There are different (but equivalent) ways how to evaluate the integrals over these delta functions, and our choice can be visualized as follows



(4.93)

Further reading

For more details about the interacting quantum Klein-Gordon field in the operator approach, see e.g. Nastase (Chapters 5 and 6), Greiner/Reinhardt (Chapters 8 and 9) and Maggiore (Chapter 5) from our reading list in Section 1.3.

Chapter 5

Free quantum Dirac field

This chapter studies the canonical quantization of the free Dirac field, which leads to fermionic spin $\frac{1}{2}$ particles and their antiparticles.

5.1 Clifford algebra and spinor representation

Different types of relativistic fields are characterized by their transformation behavior under proper and orthochronous Poincaré transformations $x' = \Lambda x + b$. For instance, we have seen in Example 2.4 that a scalar field $\Phi(x)$ transforms under active Poincaré transformations according to

$$T \,:\, \Phi(x) \longmapsto T\Phi(x) = \Phi(\Lambda^{-1}(x-b)) \tag{5.1}$$

and, more generally, a (p,q)-tensor field $A^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q}(x)$ transforms according to

$$T \,:\, A^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q}(x) \longmapsto (TA)^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_q}(x) = \Lambda^{\mu_1}{}_{\rho_1}\cdots\Lambda^{\mu_p}{}_{\rho_p}\Lambda_{\nu_1}{}^{\sigma_1}\cdots\Lambda_{\nu_q}{}^{\sigma_q}\, A^{\rho_1\cdots\rho_p}_{\sigma_1\cdots\sigma_q}(\Lambda^{-1}(x-b)) \quad . \tag{5.2}$$

These are particular cases of the general transformation law

$$T \,:\, \phi_a(x) \longmapsto (T\phi)_a(x) = D[\Lambda]_a{}^c\, \phi_c(\Lambda^{-1}(x-b)) \tag{5.3}$$

for a multicomponent field $\phi_a(x)$, where D is a *representation* of the group $\mathsf{SO}_0(d-1,1)$ of proper and orthochronous Lorentz transformations, i.e. $D[\Lambda\,\Lambda']_a{}^c = D[\Lambda]_a{}^b\, D[\Lambda']_b{}^c$ is multiplicative and it preserves the identity element $D[1]_a{}^b = \delta^b_a$.

You might now ask the question: Do (p,q)-tensor fields cover all possible cases of the general transformation law (5.3)? Mathematically, this amounts to classifying the representations D of the proper and orthochronous Lorentz group $\mathsf{SO}_0(d-1,1)$, which is a problem that has been solved. Due to time constraints, we shall not discuss this classification in detail and I refer you to e.g. Maggiore (Chapter 2) for the case of $d=4$ dimensions, which is the most relevant one for physics. The key observation is that, in addition to the (p,q)-tensor representations of $\mathsf{SO}_0(d-1,1)$, there exist spinor representations of the closely related *spin group* $\mathsf{Spin}(d-1,1)$, which is a double cover of $\mathsf{SO}_0(d-1,1)$. These are the relativistic analogs of the half-integer spin representations of the 3-dimensional rotation group $\mathsf{SO}(3)$, or more precisely of its double cover $\mathsf{Spin}(3) \cong \mathsf{SU}(2)$, which you have previously seen in quantum mechanics.

To get our hands on such spinor representations, it is useful to work, as illustrated in Example 2.10, with infinitesimal Lorentz transformations. Let us recall and further expand on this point.

The infinitesimal Lorentz transformations are characterized by real $d \times d$-matrices $\omega^\mu{}_\nu$, whose index lowering is antisymmetric $\omega_{\mu\nu} = -\omega_{\nu\mu}$. The passage to finite Lorentz transformations is via matrix exponentiation $\Lambda = e^\omega$. It is useful to factorize

$$\omega = \frac{\mathrm{i}}{2}\, \omega_{\mu\nu}\, J^{\nu\mu} \tag{5.4}$$

into the parameters $\omega_{\mu\nu} = -\omega_{\nu\mu}$ and the generators $J^{\nu\mu} = -J^{\mu\nu}$ of the infinitesimal transformations. Taking the ${}^\rho{}_\sigma$ entry on both sides of the matrix equation (5.4), one finds that the generator $J^{\nu\mu}$ is represented by the matrix

$$(J^{\nu\mu})^\rho{}_\sigma = \mathrm{i}\,(\eta^{\nu\rho}\,\delta^\mu_\sigma - \eta^{\mu\rho}\,\delta^\nu_\sigma) \quad . \tag{5.5}$$

These generators satisfy the following commutation relations

$$[J^{\mu\nu}, J^{\rho\sigma}] = \mathrm{i}\,(\eta^{\nu\rho}\,J^{\mu\sigma} - \eta^{\mu\rho}\,J^{\nu\sigma} - \eta^{\nu\sigma}\,J^{\mu\rho} + \eta^{\mu\sigma}\,J^{\nu\rho}) \quad , \tag{5.6}$$

which characterize the behavior of infinitesimal Lorentz transformations. In mathematical terminology, the infinitesimal symmetry generators and their commutation relations form a so-called *Lie algebra*, which is why (5.6) is often called the *Lorentz Lie algebra*. The matrix representation of the generators given in (5.5) is the so-called vector representation of the Lorentz Lie algebra; this terminology makes sense because (5.5) describes the infinitesimal Lorentz transformations $(\delta_\omega A)^\rho = \frac{\mathrm{i}}{2}\,\omega_{\mu\nu}\,(J^{\nu\mu})^\rho{}_\sigma\, A^\sigma = \omega^\rho{}_\sigma\, A^\sigma$ of tangent vectors $A^\mu \in \mathbb{R}^d$.

The spinor representation is a different representation of the Lorentz Lie algebra (5.6) that we shall now describe. To construct this representation, one starts from a more fundamental object, the so-called *Clifford algebra*. This is the complex algebra generated by symbols γ^μ, for $\mu = 0, 1, \ldots, d-1$, that satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,\eta^{\mu\nu} \tag{5.7}$$

involving on the right-hand side the inverse Minkowski metric $\eta^{\mu\nu}$. With some efforts that go beyond the scope of this module, one can show that, up to equivalence given by change vector space bases, there exists a unique irreducible complex representation of the Clifford algebra. This means that we can and we will, without loss of generality, think of the gamma-matrices γ^μ as complex $N \times N$-matrices acting on a complex vector space $\mathbb{C}^N$, whose dimension $N = 2^{\lfloor \frac{d}{2} \rfloor}$ is determined by the spacetime dimension. This vector space $\mathbb{C}^N$ is called the space of *Dirac spinors*. The relevance of the Clifford algebra is that the appropriately normalized commutators of the gamma-matrices

$$S^{\mu\nu} := \frac{\mathrm{i}}{4}\,[\gamma^\mu, \gamma^\nu] \tag{5.8}$$

define a representation of the Lorentz Lie algebra, i.e.

$$[S^{\mu\nu}, S^{\rho\sigma}] = \mathrm{i}\,(\eta^{\nu\rho}\,S^{\mu\sigma} - \eta^{\mu\rho}\,S^{\nu\sigma} - \eta^{\nu\sigma}\,S^{\mu\rho} + \eta^{\mu\sigma}\,S^{\nu\rho}) \quad . \tag{5.9}$$

To prove this claim, let us first note that, using the Clifford algebra relations (5.7), we can write

$$S^{\mu\nu} = \frac{\mathrm{i}}{4}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) = \frac{\mathrm{i}}{4}\left(2\,\gamma^\mu\gamma^\nu - \{\gamma^\nu,\gamma^\mu\}\right) = \frac{\mathrm{i}}{2}\,\gamma^\mu\gamma^\nu - \frac{\mathrm{i}}{2}\,\eta^{\mu\nu} \quad . \tag{5.10}$$

Using again the Clifford algebra relations, we compute

$$\begin{aligned}
[S^{\mu\nu},\gamma^\rho] &= \frac{\mathrm{i}}{2}\,[\gamma^\mu\,\gamma^\nu,\gamma^\rho] = \frac{\mathrm{i}}{2}\left(\gamma^\mu\,\gamma^\nu\,\gamma^\rho - \gamma^\rho\,\gamma^\mu\,\gamma^\nu\right)\\
&= \frac{\mathrm{i}}{2}\left(\gamma^\mu\,\gamma^\nu\,\gamma^\rho + \gamma^\mu\,\gamma^\rho\,\gamma^\nu - \{\gamma^\rho,\gamma^\mu\}\,\gamma^\nu\right)\\
&= \frac{\mathrm{i}}{2}\left(\gamma^\mu\,\gamma^\nu\,\gamma^\rho - \gamma^\mu\,\gamma^\nu\,\gamma^\rho + \gamma^\mu\,\{\gamma^\rho,\gamma^\nu\} - \{\gamma^\rho,\gamma^\mu\}\,\gamma^\nu\right)\\
&= -\mathrm{i}\left(\eta^{\mu\rho}\,\gamma^\nu - \eta^{\nu\rho}\,\gamma^\mu\right) = -(J^{\mu\nu})^\rho{}_\sigma\,\gamma^\sigma \quad ,
\end{aligned} \tag{5.11}$$

which together with the Leibniz rule for the commutator implies (5.9) by a short calculation

$$\begin{aligned}
[S^{\mu\nu},S^{\rho\sigma}] &= \frac{\mathrm{i}}{2}\,[S^{\mu\nu},\gamma^\rho\,\gamma^\sigma]\\
&= \frac{\mathrm{i}}{2}\left([S^{\mu\nu},\gamma^\rho]\,\gamma^\sigma + \gamma^\rho\,[S^{\mu\nu},\gamma^\sigma]\right)\\
&= -\frac{\mathrm{i}^2}{2}\left(\eta^{\mu\rho}\,\gamma^\nu\,\gamma^\sigma - \eta^{\nu\rho}\,\gamma^\mu\,\gamma^\sigma + \eta^{\mu\sigma}\,\gamma^\rho\,\gamma^\nu - \eta^{\nu\sigma}\,\gamma^\rho\,\gamma^\mu\right)\\
&= \mathrm{i}\left(\eta^{\nu\rho}\,S^{\mu\sigma} - \eta^{\mu\rho}\,S^{\nu\sigma} - \eta^{\nu\sigma}\,S^{\mu\rho} + \eta^{\mu\sigma}\,S^{\nu\rho}\right) \quad .
\end{aligned} \tag{5.12}$$

The representation (5.8) of the Lorentz Lie algebra on $\mathbb{C}^N$ is called the *Dirac spinor representation*. The action of a finite Lorentz transformation on such spinors $\psi \in \mathbb{C}^N$ is given by exponentiation

$$\psi \;\longmapsto\; \psi' := e^{\frac{\mathrm{i}}{2}\omega_{\mu\nu}S^{\nu\mu}}\psi \quad . \tag{5.13}$$

A slightly unpleasant feature of the Dirac spinor representation is that it is *not* a unitary representation. (In fact, the Lorentz Lie algebra (5.6) does not admit any finite-dimensional unitary representations.) This means that the standard complex inner product $\langle\psi,\tilde\psi\rangle := \psi^\dagger\,\tilde\psi$ on $\mathbb{C}^N$ is *not* invariant under the Lorentz transformations (5.13), hence it cannot be used later to build a Poincaré invariant action functional for the Dirac field. This issue can be traced back to the Clifford relations (5.7), which can be written equivalently for the individual gamma-matrices as

$$(\gamma^0)^2 \;=\; -1 \quad , \quad (\gamma^i)^2 \;=\; 1 \quad , \quad \gamma^\mu\,\gamma^\nu \;=\; -\gamma^\nu\,\gamma^\mu \quad (\text{for } \mu\neq\nu) \quad . \tag{5.14}$$

Because γ^0 squares to -1, this matrix cannot be chosen to be Hermitian, but it turns out that it can be chosen to be anti-Hermitian. The other γ^i's can be chosen to be Hermitian matrices, which allows us to demand the following (anti-)Hermiticity conditions

$$(\gamma^0)^\dagger = -\gamma^0 \quad , \quad (\gamma^i)^\dagger = \gamma^i \quad . \tag{5.15}$$

Using the Clifford algebra relations in the form of (5.14), we can write these (anti)-Hermiticity conditions in closed form as

$$(\gamma^\mu)^\dagger = \gamma^0\,\gamma^\mu\,\gamma^0 = -(\mathrm{i}\gamma^0)\,\gamma^\mu\,(\mathrm{i}\gamma^0) \quad . \tag{5.16}$$

The second expression is more useful because the matrix $(\mathrm{i}\gamma^0)$ has the following pleasant properties

$$(\mathrm{i}\gamma^0)^2 = 1 \quad , \qquad (\mathrm{i}\gamma^0)^\dagger = (\mathrm{i}\gamma^0) \quad , \tag{5.17}$$

which we will frequently use in our calculations below. From (5.16) one immediately deduces that the generators (5.8) satisfy

$$(S^{\mu\nu})^\dagger = -\frac{\mathrm{i}}{4}\,[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger] = -(\mathrm{i}\gamma^0)\,S^{\nu\mu}\,(\mathrm{i}\gamma^0) = (\mathrm{i}\gamma^0)\,S^{\mu\nu}\,(\mathrm{i}\gamma^0) \quad . \tag{5.18}$$

At the level of finite transformations, one then finds that

$$\left(e^{\frac{\mathrm{i}}{2}\omega_{\mu\nu}\,S^{\nu\mu}}\right)^\dagger = (\mathrm{i}\gamma^0)\; e^{-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\,S^{\nu\mu}}\; (\mathrm{i}\gamma^0) \quad , \tag{5.19}$$

which shows our claim that (5.13) is not a unitary representation. This expression however motivates the definition of the *Dirac adjoint*

$$\overline{\psi} \;:=\; \psi^\dagger\,(\mathrm{i}\gamma^0) \tag{5.20}$$

and its associated *Dirac inner product*

$$\overline{\psi}\,\tilde{\psi} \;:=\; \psi^\dagger\,(\mathrm{i}\gamma^0)\,\tilde{\psi} \quad , \tag{5.21}$$

which have the following pleasant Lorentz transformation properties

$$\overline{\psi}' \;=\; \overline{\psi}\,e^{-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\,S^{\nu\mu}} \quad , \qquad \overline{\psi}'\,\tilde{\psi}' \;=\; \overline{\psi}\,\tilde{\psi} \quad . \tag{5.22}$$

In particular, in contrast to the standard complex inner product $\psi^\dagger\,\tilde{\psi}$, the Dirac inner product is Lorentz invariant. To conclude our discussion of the Dirac spinor representation, we note that the identity in (5.11) exponentiates to

$$e^{\frac{\mathrm{i}}{2}\omega_{\mu\nu}\,S^{\nu\mu}}\,\gamma^\rho e^{-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\,S^{\nu\mu}} = \left(e^{-\omega}\right)^\rho{}_\sigma\,\gamma^\sigma \quad . \tag{5.23}$$

This identity expresses that the gamma-matrices intertwine between the spinor and vector representations of the Lorentz group. In particular, it implies that the tuple of complex numbers $\overline{\psi}\,\gamma^\rho\,\tilde{\psi}$, for $\rho = 0, \ldots, d-1$, transforms in the vector representation, i.e.

$$\overline{\psi}'\,\gamma^\rho\,\tilde{\psi}' = \overline{\psi}\,e^{-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\,S^{\nu\mu}}\,\gamma^\rho\,e^{\frac{\mathrm{i}}{2}\omega_{\mu\nu}\,S^{\nu\mu}}\,\tilde{\psi} = \left(e^{\omega}\right)^\rho{}_\sigma\,\overline{\psi}\,\gamma^\sigma\,\tilde{\psi} = \Lambda^\rho{}_\sigma\,\overline{\psi}\,\gamma^\sigma\,\tilde{\psi} \quad . \tag{5.24}$$

These are now all the ingredients we need to introduce the classical Dirac field in the next section.

Explicit formulas for gamma-matrices and spinors in $d = 4$ dimensions:

To get a better understanding of the concept of Dirac spinors, we shall spell out the case of $d = 4$ dimensional Dirac spinors in a fully explicit form. The space of Dirac spinors in $d = 4$ is given by

the $N = 2^{\lfloor \frac{d}{2} \rfloor} = 4$-dimensional complex vector space $\mathbb{C}^4$. A choice of representation for the gamma-matrices, which we recall is unique up to equivalence, is given by the following block 4×4-matrices

$$\gamma^0 = -\mathrm{i} \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix} \quad , \qquad \gamma^i = -\mathrm{i} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad , \tag{5.25}$$

where $1_{2\times 2}$ denotes the 2×2 identity matrix and σ^i are the three Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{5.26}$$

Recall that the Pauli matrices satisfy the following multiplication property

$$\sigma^i \, \sigma^j = \delta^{ij} + \mathrm{i} \, \epsilon^{ijk} \, \sigma^k \quad , \tag{5.27a}$$

which implies in particular the commutation and anticommutation relations

$$[\sigma^i, \sigma^j] = 2 \, \mathrm{i} \, \epsilon^{ijk} \, \sigma^k \quad , \qquad \{\sigma^i, \sigma^j\} = 2 \, \delta^{ij} \quad . \tag{5.27b}$$

Using these identities, one easily checks that (5.25) satisfies the Clifford algebra relations (5.7). For the Lie algebra generators (5.9), one finds

$$S^{0i} = -S^{i0} = \frac{\mathrm{i}}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \tag{5.28a}$$

and

$$S^{ij} = -\frac{1}{2} \begin{pmatrix} \epsilon^{ijk} \, \sigma^k & 0 \\ 0 & \epsilon^{ijk} \, \sigma^k \end{pmatrix} \quad . \tag{5.28b}$$

Exponentiating these generators, we observe that Dirac spinors transform under Lorentz boosts according to

$$\psi \;\longmapsto\; \psi' = \begin{pmatrix} e^{\frac{\omega_{0i}}{2} \, \sigma^i} & 0 \\ 0 & e^{-\frac{\omega_{0i}}{2} \, \sigma^i} \end{pmatrix} \psi \tag{5.29a}$$

and under spatial rotations according to

$$\psi \;\longmapsto\; \psi' = \begin{pmatrix} e^{\frac{\mathrm{i}}{4} \omega_{ij} \, \epsilon^{ijk} \, \sigma^k} & 0 \\ 0 & e^{\frac{\mathrm{i}}{4} \omega_{ij} \, \epsilon^{ijk} \, \sigma^k} \end{pmatrix} \psi \quad . \tag{5.29b}$$

Example 5.1 (Illustration why Dirac spinors are really spin $\frac{1}{2}$)**.** The nonrelativistic spinors you know from quantum mechanics have the characteristic feature that rotating them by 2π doesn't give back the original spinor but minus the spinor. This phenomenon is linked to the fact that

the spin groups are defined as double covers of the special orthogonal groups. The same feature is present for our relativistic Dirac spinors. Consider without loss of generality a spatial rotation in the x^1/x^2-plane, i.e. around the x^3-axis. The relevant parameters $\omega_{\mu\nu}$ for such transformation are given by $\omega_{12} = -\omega_{21} = \alpha$ and otherwise $\omega_{\mu\nu} = 0$. The vector representation for these parameters is given by

$$\omega = \frac{\mathrm{i}}{2}\,\omega_{\mu\nu}\,J^{\nu\mu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad e^{\omega} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) & 0 \\ 0 & -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \tag{5.30}$$

hence the parameter α defines the rotation angle. Using (5.29), we find that the corresponding transformation on Dirac spinors is given by

$$e^{\frac{\mathrm{i}}{2}\omega_{\mu\nu}\,S^{\nu\mu}} = \begin{pmatrix} e^{\frac{\mathrm{i}}{2}\alpha\,\sigma^3} & 0 \\ 0 & e^{\frac{\mathrm{i}}{2}\alpha\,\sigma^3} \end{pmatrix} = \begin{pmatrix} e^{\frac{\mathrm{i}\,\alpha}{2}} & 0 & 0 & 0 \\ 0 & e^{-\frac{\mathrm{i}\,\alpha}{2}} & 0 & 0 \\ 0 & 0 & e^{\frac{\mathrm{i}\,\alpha}{2}} & 0 \\ 0 & 0 & 0 & e^{-\frac{\mathrm{i}\,\alpha}{2}} \end{pmatrix} \quad . \tag{5.31}$$

In particular, rotating a Dirac spinor ψ by an angle $\alpha = 2\pi$, we find

$$\psi'_{\alpha=2\pi} \;=\; -\psi \quad , \tag{5.32}$$

hence Dirac spinors behave indeed as spin $\frac{1}{2}$ objects.

We conclude this section by making an observation that will become relevant later when we discuss the standard model of particle physics. From our explicit Lorentz transformation formulas (5.29) for Dirac spinors, we immediately see that the Dirac spinor representation in $d = 4$ is reducible. Indeed, writing

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{5.33}$$

in terms of 2-component spinors $\psi_L \in \mathbb{C}^2$ and $\psi_R \in \mathbb{C}^2$, we see that Lorentz transformations don't mix among ψ_L and ψ_R because (5.29) is block diagonal. The 2-component spinors $\psi_{L/R} \in \mathbb{C}^2$ are called the *left/right handed Weyl (or chiral) spinors*. The existence of Weyl spinors is a feature linked to the fact that the spacetime dimension $d = 4$ is even. The Weyl spinors can be defined more intrinsically in terms of the Clifford algebra by introducing the product of gamma-matrices

$$\gamma_5 \;:=\; -\mathrm{i}\,\gamma^0\,\gamma^1\,\gamma^2\,\gamma^3 \quad , \tag{5.34}$$

which in our representation (5.25) reads as

$$\gamma_5 \;=\; \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix} \quad . \tag{5.35}$$

Introducing the projectors

$$P_L := \frac{1+\gamma_5}{2} = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & 0 \end{pmatrix} \quad , \qquad P_R := \frac{1-\gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1_{2\times 2} \end{pmatrix} \quad , \tag{5.36}$$

one obtains the following more intrinsic characterization of left/right handed spinors

$$\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad P_R \psi = 0 \quad , \tag{5.37a}$$

$$\psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix} \quad \Longleftrightarrow \quad P_L \psi = 0 \quad . \tag{5.37b}$$

5.2 Classical Dirac field

A *Dirac field* is defined as a Dirac spinor-valued function $\Psi : \mathbb{R}^d \to \mathbb{C}^N$, $x \mapsto \Psi(x)$ on the Minkowski spacetime $\mathbb{R}^d$. The transformation law of a Dirac field under active Poincaré transformations is then given by

$$T_{(\omega,b)} : \Psi(x) \;\longmapsto\; T_{(\omega,b)}\Psi(x) \;=\; e^{\frac{\mathrm{i}}{2}\omega_{\mu\nu}S^{\nu\mu}}\, \Psi\big(e^{-\omega}(x-b)\big) \quad , \tag{5.38}$$

where $S^{\nu\mu}$ are the Lorentz generators in the Dirac spinor representation (5.8). Using the Dirac adjoint from (5.20), we introduce the Dirac action functional

$$\boxed{S_{\mathrm{Dirac}}[\Psi,\overline{\Psi}] := -\int_{\mathbb{R}^d} \overline{\Psi}\,(\not{\partial} + m)\,\Psi\,\mathrm{d}x := -\int_{\mathbb{R}^d} \overline{\Psi}\,(\gamma^\mu\,\partial_\mu + m)\,\Psi\,\mathrm{d}x \quad ,} \tag{5.39}$$

where $m > 0$ is a positive parameter that will be interpreted later as the mass. (To simplify this section, we consider only the massive Dirac field because the massless case $m = 0$ is slightly more subtle.) Here and in the following we shall use the convenient *Feynman slash notation*

$$\boxed{\not{B} := \gamma^\mu\, B_\mu} \tag{5.40}$$

to denote the contraction of a covector B_μ with the gamma-matrices γ^μ. Using (5.22) and (5.24), one easily checks that the Dirac action is invariant under the Poincaré transformations given in (5.38). Using also the anti-Hermiticity properties (5.16), one shows that the Dirac action is, as required, real-valued

$$\begin{aligned} S_{\mathrm{Dirac}}[\Psi,\overline{\Psi}]^\dagger &= -\int_{\mathbb{R}^d} \Big(\Psi^\dagger\,(\mathrm{i}\gamma^0)\,\gamma^\mu\,\partial_\mu\Psi + m\,\Psi^\dagger\,(\mathrm{i}\gamma^0)\,\Psi\Big)^\dagger\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^d} \Big(-\partial_\mu\Psi^\dagger\,(\mathrm{i}\gamma^0)\,\gamma^\mu\,\Psi + m\,\Psi^\dagger\,(\mathrm{i}\gamma^0)\,\Psi\Big)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^d} \Big(\Psi^\dagger\,(\mathrm{i}\gamma^0)\,\gamma^\mu\,\partial_\mu\Psi + m\,\Psi^\dagger\,(\mathrm{i}\gamma^0)\,\Psi\Big)\,\mathrm{d}x = S_{\mathrm{Dirac}}[\Psi,\overline{\Psi}] \quad , \end{aligned} \tag{5.41}$$

where in the third step we have integrated by parts.

The Euler-Lagrange equations associated with the Dirac action are given by

$$(\not{\partial} + m)\Psi \;=\; 0 \quad , \qquad -\partial_\mu \overline{\Psi}\,\gamma^\mu + m\,\overline{\Psi} \;=\; 0 \quad . \tag{5.42}$$

Note that the second equation is simply the Dirac adjoint of the first equation

$$0 = \overline{(\gamma^\mu \partial_\mu + m)\Psi} = \partial_\mu \Psi^\dagger\,(\gamma^\mu)^\dagger\,(\mathrm{i}\gamma^0) + m\,\overline{\Psi} = -\partial_\mu \overline{\Psi}\,\gamma^\mu + m\,\overline{\Psi} \quad . \tag{5.43}$$

The equation $(\not{\partial} + m)\Psi = 0$ is the famous *Dirac equation*, which was originally discovered by Dirac in his search for a "square root" of the Klein-Gordon operator $-\partial^2 + m^2$. Indeed, multiplying both sides of the Dirac equation by the "wrong sign Dirac operator" $(-\not{\partial} + m)$, we observe with some Clifford algebra (5.7) that

$$\begin{aligned} 0 &= (-\not{\partial} + m)(\not{\partial} + m)\Psi = -\not{\partial}^2\,\Psi + m^2\,\Psi = -\gamma^\nu\gamma^\mu\,\partial_\nu\partial_\mu\Psi + m^2\,\Psi \\ &= -\frac{1}{2}\{\gamma^\nu,\gamma^\mu\}\,\partial_\nu\partial_\mu\Psi + m^2\,\Psi = -\eta^{\nu\mu}\partial_\nu\partial_\mu\Psi + m^2\,\Psi = (-\partial^2 + m^2)\Psi \end{aligned} \tag{5.44}$$

gives the Klein-Gordon equation. This implies in particular that every solution Ψ of the Dirac equation satisfies automatically the Klein-Gordon equation $(-\partial^2 + m^2)\Psi = 0$ too.

Finding solutions of the Dirac equation is more involved than in the case of the Klein-Gordon equation, because the Dirac spinor $\Psi(x) \in \mathbb{C}^N$ is a multicomponent field and the Dirac equation (5.42) imposes nontrivial conditions among the individual components. Since the Dirac equation is linear, we can write any solution $\Psi(x)$ as a superposition of the following two types of plane waves

$$\Psi(x) = u(\mathbf{k})\,e^{\mathrm{i}\,k\,x} \quad , \qquad \Psi(x) = v(\mathbf{k})\,e^{-\mathrm{i}\,k\,x} \quad , \tag{5.45}$$

where $k = (\omega_{\mathbf{k}}, \mathbf{k}) \in \mathbb{R}^d$ is a relativistic on-shell momentum with positive energy $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. Note that the on-shell condition follows from the fact that $\Psi(x)$ satisfies the Klein-Gordon equation, which for plane waves enforces $k^2 + m^2 = 0$. Inserting these plane waves into the Dirac equation, we find that the spinor polarizations $u(\mathbf{k}), v(\mathbf{k}) \in \mathbb{C}^N$ must satisfy the conditions

$$(\mathrm{i}\not{k} + m)\,u(\mathbf{k}) = 0 \quad , \qquad (-\mathrm{i}\not{k} + m)\,v(\mathbf{k}) = 0 \quad . \tag{5.46}$$

To understand these equations better, let us introduce the linear maps

$$P^\pm(\mathbf{k}) \;:=\; \frac{1}{2m}\,(\pm\,\mathrm{i}\not{k} + m) \tag{5.47}$$

on the space of Dirac spinors. The spinor polarization equations then read equivalently as

$$P^+(\mathbf{k})\,u(\mathbf{k}) = 0 \quad , \qquad P^-(\mathbf{k})\,v(\mathbf{k}) = 0 \quad . \tag{5.48}$$

By direct inspection, one observes that these linear maps satisfy the following identities

$$P^+(\mathbf{k}) + P^-(\mathbf{k}) \;=\; 1 \quad , \quad P^\pm(\mathbf{k})\,P^\pm(\mathbf{k}) \;=\; P^\pm(\mathbf{k}) \quad , \quad P^\pm(\mathbf{k})\,P^\mp(\mathbf{k}) \;=\; 0 \quad , \tag{5.49}$$

which means that they are complementary projectors. This allows us to decompose the space of Dirac spinors into a direct sum

$$\mathbb{C}^N = \mathrm{Im}\big(P^-(\mathbf{k})\big) \oplus \mathrm{Im}\big(P^+(\mathbf{k})\big) = \mathrm{Ker}\big(P^+(\mathbf{k})\big) \oplus \mathrm{Ker}\big(P^-(\mathbf{k})\big) \quad , \tag{5.50}$$

where we use that

$$\mathrm{Im}\big(P^\pm(\mathbf{k})\big) \;=\; \mathrm{Ker}\big(P^\mp(\mathbf{k})\big) \quad . \tag{5.51}$$

Lemma 5.2. *The decomposition (5.50) of the space of Dirac spinors is orthogonal with respect to the Dirac inner product. Furthermore, the Dirac inner product is positive definite on the first summand* $\mathrm{Ker}\big(P^+(\mathbf{k})\big)$ *and it is negative definite on the second summand* $\mathrm{Ker}\big(P^-(\mathbf{k})\big)$*. In formulas, all this means that, for any nonzero* $u(\mathbf{k}) \in \mathrm{Ker}\big(P^+(\mathbf{k})\big)$ *and* $v(\mathbf{k}) \in \mathrm{Ker}\big(P^-(\mathbf{k})\big)$*, we have*

$$\overline{u(\mathbf{k})}\, v(\mathbf{k}) \;=\; 0 \;=\; \overline{v(\mathbf{k})}\, u(\mathbf{k}) \quad , \quad \overline{u(\mathbf{k})}\, u(\mathbf{k}) \;>\; 0 \quad , \quad \overline{v(\mathbf{k})}\, v(\mathbf{k}) \;<\; 0 \quad . \tag{5.52}$$

Proof. Let us first note that the Dirac adjoint of the projector $P^\pm(\mathbf{k})$ is given by

$$\overline{P^\pm(\mathbf{k})\,\psi} = \psi^\dagger \left(P^\pm(\mathbf{k})\right)^\dagger (\mathrm{i}\gamma^0) = \psi^\dagger \,(\mathrm{i}\gamma^0)\, P^\pm(\mathbf{k}) = \overline{\psi}\, P^\pm(\mathbf{k}) \quad . \tag{5.53}$$

Using (5.50), we can write $u(\mathbf{k}) = P^-(\mathbf{k})\,\psi$ and $v(\mathbf{k}) = P^+(\mathbf{k})\,\tilde{\psi}$ for some spinors $\psi, \tilde{\psi} \in \mathbb{C}^N$, from which we deduce that

$$\overline{u(\mathbf{k})}\, v(\mathbf{k}) = \overline{P^-(\mathbf{k})\psi}\, P^+(\mathbf{k})\tilde{\psi} = \overline{\psi}\, P^-(\mathbf{k})\, P^+(\mathbf{k})\, \tilde{\psi} = 0 \tag{5.54a}$$

and similarly

$$\overline{v(\mathbf{k})}\, u(\mathbf{k}) = \overline{P^+(\mathbf{k})\tilde{\psi}}\, P^-(\mathbf{k})\psi = \overline{\tilde{\psi}}\, P^+(\mathbf{k})\, P^-(\mathbf{k})\, \psi = 0 \quad . \tag{5.54b}$$

To prove the statement about the definiteness of the Dirac inner product, we use that the Dirac inner product is Lorentz invariant. This means that we can transform, without loss of generality, to the rest frame where $\mathbf{k} = \mathbf{0}$ and consequently $k = (m, \mathbf{0})$. The projectors in the rest frame simplify to $P^\pm(\mathbf{0}) = \frac{1}{2}(1 \mp \mathrm{i}\,\gamma^0)$ and they enjoy the following useful properties

$$\left(P^\pm(\mathbf{0})\right)^\dagger = P^\pm(\mathbf{0}) \quad , \quad (\mathrm{i}\gamma^0)\, P^\pm(\mathbf{0}) = \mp\, P^\pm(\mathbf{0}) \quad . \tag{5.55}$$

Writing again $u(\mathbf{0}) = P^-(\mathbf{0})\,\psi$ and $v(\mathbf{0}) = P^+(\mathbf{0})\,\tilde{\psi}$ for some spinors $\psi, \tilde{\psi} \in \mathbb{C}^N$, we find for the Dirac inner products

$$\overline{u(\mathbf{0})}\, u(\mathbf{0}) = \psi^\dagger \,(\mathrm{i}\gamma^0)\, P^-(\mathbf{0})\, P^-(\mathbf{0})\, \psi = \left(P^-(\mathbf{0})\,\psi\right)^\dagger \left(P^-(\mathbf{0})\,\psi\right) = \left\| P^-(\mathbf{0})\,\psi \right\|^2 > 0 \tag{5.56a}$$

and similarly

$$\overline{v(\mathbf{0})}\, v(\mathbf{0}) = \tilde{\psi}^\dagger \,(\mathrm{i}\gamma^0)\, P^+(\mathbf{0})\, P^+(\mathbf{0})\, \tilde{\psi} = -\left(P^+(\mathbf{0})\,\tilde{\psi}\right)^\dagger \left(P^+(\mathbf{0})\,\tilde{\psi}\right) = -\left\| P^+(\mathbf{0})\,\tilde{\psi} \right\|^2 < 0 \quad , \tag{5.56b}$$

where in the last steps we recognized the standard norm $\|\cdot\|$ on $\mathbb{C}^N$. □

The relevance of this lemma is that it allows us to choose vector space bases

$$\Big\{u^s(\mathbf{k}) \in \mathrm{Ker}(P^+(\mathbf{k}))\Big\} \quad , \qquad \Big\{v^s(\mathbf{k}) \in \mathrm{Ker}(P^-(\mathbf{k}))\Big\} \tag{5.57a}$$

that are orthonormal, with respect to the indefinite Dirac inner product, according to

$$\overline{u^s(\mathbf{k})}\, u^r(\mathbf{k}) \;=\; 2m\,\delta^{sr} \quad , \tag{5.57b}$$
$$\overline{v^s(\mathbf{k})}\, v^r(\mathbf{k}) \;=\; -2m\,\delta^{sr} \quad , \tag{5.57c}$$
$$\overline{u^s(\mathbf{k})}\, v^r(\mathbf{k}) \;=\; 0 \;=\; \overline{v^s(\mathbf{k})}\, u^r(\mathbf{k}) \quad . \tag{5.57d}$$

The projectors $P^{\pm}(\mathbf{k})$ can be written in terms of these bases as

$$P^+(\mathbf{k}) = -\frac{1}{2m}\sum_s v^s(\mathbf{k})\,\overline{v^s(\mathbf{k})} \quad , \qquad P^-(\mathbf{k}) = \frac{1}{2m}\sum_s u^s(\mathbf{k})\,\overline{u^s(\mathbf{k})} \quad , \tag{5.58a}$$

which, upon rearrangement, yields the so-called *spin sum identities*

$$\sum_s u^s(\mathbf{k})\,\overline{u^s(\mathbf{k})} \;=\; (-\mathrm{i}\not{k} + m) \quad , \tag{5.58b}$$
$$\sum_s v^s(\mathbf{k})\,\overline{v^s(\mathbf{k})} \;=\; -(\mathrm{i}\not{k} + m) \quad . \tag{5.58c}$$

Remark 5.3. One can derive further spinor polarization identities that will facilitate our calculations with Dirac spinors in the next sections. These identities arise from the observation that taking the ordinary adjoint of the projector $P^{\pm}(\mathbf{k})$ gives

$$(P^{\pm}(\mathbf{k}))^\dagger = \frac{1}{2m}(\mp\mathrm{i}\gamma^0\omega_{\mathbf{k}} \pm \mathrm{i}\gamma^i k_i + m)^\dagger = \frac{1}{2m}(\mp\mathrm{i}\gamma^0\omega_{\mathbf{k}} \mp \mathrm{i}\gamma^i k_i + m) = P^{\pm}(-\mathbf{k}) \quad , \tag{5.59}$$

where $-\mathbf{k}$ is the oppositely pointing spatial momentum. The conditions (5.48) for the spinor polarizations then translate for the ordinary adjoint spinors to

$$u^s(\mathbf{k})^\dagger\, P^+(-\mathbf{k}) \;=\; 0 \quad , \qquad v^s(\mathbf{k})^\dagger\, P^-(-\mathbf{k}) \;=\; 0 \quad . \tag{5.60}$$

Using also $P^+(-\mathbf{k})\, v^r(-\mathbf{k}) = v^r(-\mathbf{k})$ and $P^-(-\mathbf{k})\, u^r(-\mathbf{k}) = u^r(-\mathbf{k})$, we obtain the orthogonality conditions

$$u^s(\mathbf{k})^\dagger\, v^r(-\mathbf{k}) = u^s(\mathbf{k})^\dagger\, P^+(-\mathbf{k})\, v^r(-\mathbf{k}) = 0 \quad , \tag{5.61a}$$
$$v^s(\mathbf{k})^\dagger\, u^r(-\mathbf{k}) = v^s(\mathbf{k})^\dagger\, P^-(-\mathbf{k})\, u^r(-\mathbf{k}) = 0 \tag{5.61b}$$

for oppositely pointing spatial momenta. We will also need formulas for the standard inner products $u^s(\mathbf{k})^\dagger\, u^r(\mathbf{k})$ and $v^s(\mathbf{k})^\dagger\, v^r(\mathbf{k})$ without Dirac adjoints. These can be obtained using our projectors via the following trick

$$u^s(\mathbf{k})^\dagger\, u^r(\mathbf{k}) = \frac{1}{2}\, u^s(\mathbf{k})^\dagger\, (P^-(\mathbf{k}) + P^-(-\mathbf{k}))\, u^r(\mathbf{k}) = \frac{\omega_{\mathbf{k}}}{2m}\,\overline{u^s(\mathbf{k})}\, u^r(\mathbf{k}) + \frac{1}{2}\, u^s(\mathbf{k})^\dagger\, u^r(\mathbf{k}) \quad , \tag{5.62}$$

which after rearrangement and using that $\overline{u^s(\mathbf{k})}\, u^r(\mathbf{k}) = 2m\,\delta^{sr}$ yields

$$u^s(\mathbf{k})^\dagger\, u^r(\mathbf{k}) = 2\omega_{\mathbf{k}}\,\delta^{sr} \quad . \tag{5.63}$$

Similarly, for the v's we use

$$v^s(\mathbf{k})^\dagger\, v^r(\mathbf{k}) = \frac{1}{2}\, v^s(\mathbf{k})^\dagger\, (P^+(\mathbf{k}) + P^+(-\mathbf{k}))\, v^r(\mathbf{k}) = -\frac{\omega_{\mathbf{k}}}{2m}\, \overline{v^s(\mathbf{k})}\, v^r(\mathbf{k}) + \frac{1}{2}\, v^s(\mathbf{k})^\dagger\, v^r(\mathbf{k}) \quad , \tag{5.64}$$

which yields

$$v^s(\mathbf{k})^\dagger\, v^r(\mathbf{k}) = 2\omega_{\mathbf{k}}\,\delta^{sr} \quad . \tag{5.65}$$

Note that, in contrast to the Dirac inner product (5.57), the standard inner product is, of course, positive definite.

To conclude this section, we shall discuss briefly the Hamiltonian and conserved Noether charges of the Dirac field. More details can be found in the literature, see in particular the textbooks listed in Section 1.3.

Hamiltonian: To derive the Hamiltonian, let us take any splitting of the coordinates $x = (t, \mathbf{x}) \in \mathbb{R}^d$ of Minkowski spacetime into time and space coordinates. This allows us to write the Dirac action (5.39) as

$$S_{\text{Dirac}} = \mathrm{i} \int_{\mathbb{R}^d} \Big(\Psi^\dagger\, \dot{\Psi} - \Psi^\dagger\, \gamma^0 \gamma^i\, \partial_i \Psi - m\, \Psi^\dagger\, \gamma^0\, \Psi \Big)\, \mathrm{d}x \quad , \tag{5.66}$$

where by $\dot{\Psi} = \partial_0 \Psi = \frac{\partial}{\partial t} \Psi$ we denote again the time derivative. The canonical momenta are

$$\pi_\Psi = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = \mathrm{i}\, \Psi^\dagger \quad , \qquad \pi_{\Psi^\dagger} = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}^\dagger} = 0 \quad , \tag{5.67}$$

which gives the Hamiltonian

$$H \;=\; \mathrm{i} \int_{\mathbb{R}^{d-1}} \Big(\Psi^\dagger\, \gamma^0\, \gamma^i\, \partial_i \Psi + m\, \Psi^\dagger\, \gamma^0\, \Psi \Big)\, \mathrm{d}\mathbf{x} \quad . \tag{5.68}$$

Here and in what follows we simply write $\mathrm{i}\, \Psi^\dagger$ instead of π_Ψ for the canonical momentum, which is the usual convention in the literature.

Conserved charges: By construction, the Dirac action (5.39) is invariant under the Poincaré transformations given in (5.38). Furthermore, in analogy to the complex scalar field from Example 2.5, it has an internal continuous symmetry given by the constant complex phase rotations $T_\alpha \Psi(x) = e^{-\mathrm{i}\,\alpha}\, \Psi(x)$ and $T_\alpha \overline{\Psi}(x) = e^{\mathrm{i}\,\alpha}\, \overline{\Psi}(x)$. Noether's Theorem 2.6 then yields, via a routine calculation that you should try on your own, conserved on-shell currents and charges that are associated to these symmetries. The conserved charges associated with spacetime translations read as

$$P^\mu \;=\; -\mathrm{i} \int_{\mathbb{R}^{d-1}} \Psi^\dagger\, \partial^\mu \Psi\, \mathrm{d}\mathbf{x} \quad . \tag{5.69}$$

Rewriting the Dirac equation $(\gamma^\mu\partial_\mu + m)\Psi = 0$ as $\partial_0\Psi = \gamma^0\gamma^i\partial_i\Psi + m\,\gamma^0\Psi$, one checks that the $\mu = 0$ component of these conserved charges agrees with the Hamiltonian (5.68),

$$P^0 = \mathrm{i}\int_{\mathbb{R}^{d-1}} \Psi^\dagger\,\partial_0\Psi\,\mathrm{d}\mathbf{x} = \mathrm{i}\int_{\mathbb{R}^{d-1}} \Big(\Psi^\dagger\,\gamma^0\,\gamma^i\,\partial_i\Psi + m\,\Psi^\dagger\,\gamma^0\,\Psi\Big)\,\mathrm{d}\mathbf{x} = H \quad . \tag{5.70}$$

The conserved charges associated with Lorentz transformations read as

$$L^{\mu\nu} \;=\; -\mathrm{i}\int_{\mathbb{R}^{d-1}} \Big(\Psi^\dagger(x^\mu\,\partial^\nu - x^\nu\,\partial^\mu)\Psi - \mathrm{i}\,\Psi^\dagger S^{\mu\nu}\Psi\Big)\;\mathrm{d}\mathbf{x} \quad . \tag{5.71}$$

Comparing this with our results for the Klein-Gordon field in Example 2.10, we note that the second term involving the Lorentz generator $S^{\mu\nu}$ from (5.9) is a new feature. This term captures the internal relativistic angular momentum, a.k.a. the spin, of the Dirac field. The conserved charge associated with complex phase rotations reads as

$$Q = \int_{\mathbb{R}^{d-1}} \Psi^\dagger\,\Psi\,\mathrm{d}\mathbf{x} \quad . \tag{5.72}$$

5.3 Canonical quantization of the Dirac field

The quantization of the Dirac field can be achieved by promoting the fixed-time Dirac field $\Psi(\mathbf{x}) = \Psi(t_0,\mathbf{x})$ and its canonical momentum $\pi_\Psi(\mathbf{x}) = \mathrm{i}\,\Psi^\dagger(\mathbf{x}) = \mathrm{i}\,\Psi^\dagger(t_0,\mathbf{x})$ to operators. However, in contrast to the Klein-Gordon field discussed in Section 3.1, one can *not* use equal-time commutation relations as in (3.1) to obtain a consistent quantization of the Dirac field. The reason for this is rather deep and it is rooted in the so-called *spin-statistics theorem* of QFT. Without going too much into the details, which lie beyond the scope of this module, the main insight from this theorem is that integer spin fields must be quantized by using commutators and that half-integer spin fields must be quantized by using *anticommutators*. Recall that the anticommutator of two operators A and B is defined by $\{A,B\} := A\,B + B\,A$. Since the Dirac field is spin $\frac{1}{2}$, and hence in particular of half-integer spin, it makes perfectly sense in the light of the spin-statistics theorem to start instead of (3.1) with the following *equal-time anticommutation relations*

$$\big\{\Psi_\alpha(\mathbf{x}),\Psi_\beta(\mathbf{y})\big\} \;=\; 0 \;=\; \big\{\Psi^\dagger_\alpha(\mathbf{x}),\Psi^\dagger_\beta(\mathbf{y})\big\} \quad , \tag{5.73a}$$

$$\big\{\Psi_\alpha(\mathbf{x}),\Psi^\dagger_\beta(\mathbf{y})\big\} \;=\; \delta_{\alpha\beta}\,\delta(\mathbf{x}-\mathbf{y}) \quad , \tag{5.73b}$$

where the indices α,β are used to denote the components of Dirac spinors. Note that there is no factor of i on the right-hand side of the nontrivial anticommutation relation, because the canonical momentum differs from $\Psi^\dagger$ by a factor of i, namely $\pi_\Psi(\mathbf{x}) = \mathrm{i}\,\Psi^\dagger(\mathbf{x})$. To obtain the *Heisenberg picture Dirac field operator* $\Psi(x)$, which is an operator on spacetime $x \in \mathbb{R}^d$, one has to solve Heisenberg's equation

$$\frac{\partial}{\partial t}\Psi(x) \;=\; \mathrm{i}\,\big[H,\Psi(x)\big] \quad , \tag{5.74}$$

subject to the initial condition $\Psi(0,\mathbf{x}) = \Psi(\mathbf{x})$, where H denotes the Hamiltonian operator that is obtained by quantizing the Dirac Hamiltonian (5.68). (Normal ordering of the Hamiltonian operator

H is inessential at this point, because under the commutator we have that $[:H:,-] = [H,-]$.) Note that, in contrast to the anticommutators $\{-,-\}$ in (5.73), the Heisenberg equation involves the usual commutator $[-,-]$. Working out this commutator by using (5.68) and (5.73), one finds that

$$\frac{\partial}{\partial t}\Psi(x) \;=\; \gamma^0\,\gamma^i\,\partial_i\Psi(x) + m\,\gamma^0\,\Psi(x) \quad\Longleftrightarrow\quad (\not{\partial} + m)\Psi(x) = 0 \quad , \tag{5.75}$$

i.e. the Heisenberg picture Dirac field operator $\Psi(x)$ satisfies the Dirac equation.

Remark 5.4. The interplay between commutators and anticommutators is best understood by using the concept of a *graded commutator*. For this we declare both the Dirac field operator $\Psi(\mathbf{x})$ and its adjoint $\Psi^\dagger(\mathbf{x})$ to be *odd* operators, which we write in terms of the $\mathbb{Z}_2$-parities $|\Psi(\mathbf{x})| = 1 \in \mathbb{Z}_2$ and $|\Psi^\dagger(\mathbf{x})| = 1 \in \mathbb{Z}_2$. The identity operator 1 is declared to be *even*, written as $|1| = 0 \in \mathbb{Z}_2$. This concept of odd and even can be extended to products of operators: Given two operators A and B with parities $|A|$ and $|B|$, then their product $A\,B$ has parity $|A\,B| := (|A| + |B|) \mod 2 \in \mathbb{Z}_2$. For example, the product $\Psi^\dagger\,\Psi$ has parity $|\Psi^\dagger\Psi| = (1+1) \mod 2 = 0$, hence it is an even operator. More generally, even monomials of the Ψ's and $\Psi^\dagger$'s define even operators and odd monomials are odd operators. The Hamiltonian operator associated with (5.68) is even because it is quadratic. The graded commutator of two operators A and B with parities $|A|$ and $|B|$ is defined as

$$\big[A,B\big]_{\text{gr}} := A\,B - (-1)^{|A|\,|B|}\,B\,A \quad . \tag{5.76}$$

Note that if either A or B (or both) are even, then this is a commutator, while when both A and B are odd, the graded commutator gives the anticommutator. This means that we can write the equal-time anticommutation relations (5.73) equivalently in terms of the graded commutator as

$$\big[\Psi_\alpha(\mathbf{x}),\Psi_\beta(\mathbf{y})\big]_{\text{gr}} \;=\; 0 \;=\; \big[\Psi^\dagger_\alpha(\mathbf{x}),\Psi^\dagger_\beta(\mathbf{y})\big]_{\text{gr}} \;=\; 0 \quad , \tag{5.77a}$$

$$\big[\Psi_\alpha(\mathbf{x}),\Psi^\dagger_\beta(\mathbf{y})\big]_{\text{gr}} \;=\; \delta_{\alpha\beta}\,\delta(\mathbf{x}-\mathbf{y}) \tag{5.77b}$$

and Heisenberg's equation as

$$\frac{\partial}{\partial t}\Psi(x) \;=\; \mathrm{i}\,\big[H,\Psi(x)\big]_{\text{gr}} \quad . \tag{5.78}$$

The computation of the right-hand side of Heisenberg's equation is then easy by using the following identities for the graded commutator

$$\big[A,B\big]_{\text{gr}} \;=\; -(-1)^{|A|\,|B|}\,\big[B,A\big]_{\text{gr}} \quad , \tag{5.79a}$$

$$\big[A,B\,C\big]_{\text{gr}} \;=\; \big[A,B\big]_{\text{gr}}\,C + (-1)^{|A|\,|B|}\,B\,\big[A,C\big]_{\text{gr}} \quad , \tag{5.79b}$$

$$\big[A\,B,C\big]_{\text{gr}} \;=\; A\,\big[B,C\big]_{\text{gr}} + (-1)^{|B|\,|C|}\,\big[A,C\big]_{\text{gr}}\,B \quad , \tag{5.79c}$$

which I recommend you to confirm on your own. The graded commutator also satisfies the graded Jacobi identity

$$(-1)^{|A|\,|C|}\,\big[A,\big[B,C\big]_{\text{gr}}\big]_{\text{gr}} + (-1)^{|B|\,|A|}\,\big[B,\big[C,A\big]_{\text{gr}}\big]_{\text{gr}} + (-1)^{|C|\,|B|}\,\big[C,\big[A,B\big]_{\text{gr}}\big]_{\text{gr}} = 0 \quad , \tag{5.80}$$

which we however do not need at the moment.

To solve Heisenberg's equation for the Dirac field operator, which as we have seen above is equivalent to the Dirac equation, we can use our knowledge from Section 5.2 of all plane wave solutions. Using the Dirac spinor bases from (5.57), we can write

$$\Psi(x) = \int_{\mathbb{R}^{d-1}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \sum_s \Big(a_s(\mathbf{k})\, u^s(\mathbf{k})\, e^{\mathrm{i}\, k\, x} + b_s^\dagger(\mathbf{k})\, v^s(\mathbf{k})\, e^{-\mathrm{i}\, k\, x} \Big) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \tag{5.81a}$$

where $k = (\omega_{\mathbf{k}}, \mathbf{k}) \in \mathbb{R}^d$ is the relativistic on-shell Fourier momentum. The Dirac adjoint of this expression is given by

$$\overline{\Psi}(x) = \int_{\mathbb{R}^{d-1}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \sum_s \Big(b_s(\mathbf{k})\, \overline{v^s(\mathbf{k})}\, e^{\mathrm{i}\, k\, x} + a_s^\dagger(\mathbf{k})\, \overline{u^s(\mathbf{k})}\, e^{-\mathrm{i}\, k\, x} \Big) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad . \tag{5.81b}$$

Comparing with the case of the real Klein-Gordon quantum field (3.20), the following remarks are in order:

- In (5.81) there is a sum over all spinor polarizations and consequently there are spinor polarization labels s on the annihilation and creation operators. For a scalar field (3.20) this sum is trivial because Φ has only a single component.

- In (5.81) there are two different sets of annihilation and creation operators given by $a_s(\mathbf{k})$, $a_s^\dagger(\mathbf{k})$ and $b_s(\mathbf{k})$, $b_s^\dagger(\mathbf{k})$. This is due to the fact that $\Psi(x)$ is the quantization of a complex field. A similar feature is present for the complex Klein-Gordon field, where instead of the operator $a^\dagger(\mathbf{k})$ in (3.20) one has a $b^\dagger(\mathbf{k})$. The Hermiticity condition $\Phi(x)^\dagger = \Phi(x)$ for the real Klein-Gordon field forces the two sets of operators to coincide, i.e. $b^\dagger(\mathbf{k}) = a^\dagger(\mathbf{k})$ and $b(\mathbf{k}) = a(\mathbf{k})$.

- As a consequence of the equal-time *anti*commutation relations (5.73), the annihilation and creation operators in (5.81) have to satisfy *anti*commutation relations too. The nontrivial ones are given explicitly by

$$\big\{ a_s(\mathbf{k}), a_r^\dagger(\mathbf{q}) \big\} \;=\; \big\{ b_s(\mathbf{k}), b_r^\dagger(\mathbf{q}) \big\} \;=\; \delta_{sr}\, (2\pi)^{d-1}\, \delta(\mathbf{k} - \mathbf{q}) \quad , \tag{5.82a}$$

  while all other anticommutators are zero, i.e.

$$\big\{ a_s(\mathbf{k}), a_r(\mathbf{q}) \big\} = \big\{ a_s(\mathbf{k}), b_r(\mathbf{q}) \big\} = \big\{ a_s(\mathbf{k}), b_r^\dagger(\mathbf{q}) \big\} = 0 \quad , \tag{5.82b}$$

$$\big\{ a_s^\dagger(\mathbf{k}), a_r^\dagger(\mathbf{q}) \big\} = \big\{ a_s^\dagger(\mathbf{k}), b_r(\mathbf{q}) \big\} = \big\{ a_s^\dagger(\mathbf{k}), b_r^\dagger(\mathbf{q}) \big\} = 0 \quad , \tag{5.82c}$$

$$\big\{ b_s(\mathbf{k}), b_r(\mathbf{q}) \big\} = \big\{ b_s^\dagger(\mathbf{k}), b_r^\dagger(\mathbf{q}) \big\} = 0 \quad . \tag{5.82d}$$

  To check that (5.81) together with the anticommutation relations (5.82) really implies the equal-time anticommutation relations (5.73), one has to remember and use the spin sum identities from (5.58).

Similarly to the case of the quantum Klein-Gordon field in Section 3.3, we can build a Hilbert space $\mathcal{H}$ by introducing a vacuum state $|0\rangle \in \mathcal{H}$ that is characterized by the property of being normalized $\langle 0|0\rangle = 1$ and being annihilated by all annihilation operators, i.e.

$$a_s(\mathbf{k})|0\rangle = 0 \quad , \quad b_s(\mathbf{k})|0\rangle = 0 \quad . \tag{5.83}$$

Multiparticle states are then obtained by acting on $|0\rangle$ with suitably normalized creation operators. Since there are two sets of such creation operators, namely $a_s^\dagger(\mathbf{k})$ and $b_s^\dagger(\mathbf{k})$, which further depend on a choice of spinor polarization s, such multiparticle states have more labels than only a relativistic on-shell momentum $k = (\omega_{\mathbf{k}}, \mathbf{k}) \in \mathbb{R}^d$. We denote such states by

$$\begin{aligned} &\big|(k_1, s_1, +), \dots, (k_l, s_l, +), (k_{l+1}, s_{l+1}, -), \dots, (k_n, s_n, -)\big\rangle := \\ &\qquad \sqrt{2\omega_{\mathbf{k}_1}} \cdots \sqrt{2\omega_{\mathbf{k}_n}}\; a_{s_1}^\dagger(\mathbf{k}_1) \cdots a_{s_l}^\dagger(\mathbf{k}_l)\, b_{s_{l+1}}^\dagger(\mathbf{k}_{l+1}) \cdots b_{s_n}^\dagger(\mathbf{k}_n)|0\rangle \quad , \end{aligned} \tag{5.84}$$

where we attach a $\pm$ sign to distinguish between a-type and b-type particles. As a direct consequence of the anticommutation relations (5.82), we observe that exchanging any neighboring pairs of labels in such states gives a minus sign, hence the particles associated to the Dirac quantum field describe fermions.

To study further properties of these particles, and in particular to figure out what's the difference between a-type and b-type particles, we use the conserved charges that we have computed in (5.69), (5.71) and (5.72). Using (5.81) and the additional spinor polarization identities from Remark 5.3, one can express these conserved charges in terms of annihilation and creation operators. For the relativistic momentum (5.69) one then finds

$$P^\mu = \sum_s \int_{\mathbb{R}^{d-1}} k^\mu \left(a_s^\dagger(\mathbf{k})\, a_s(\mathbf{k}) - b_s(\mathbf{k})\, b_s^\dagger(\mathbf{k}) \right) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad . \tag{5.85}$$

The normal ordering of products of odd operators can be defined similarly as in Definition 3.1, with the additional rule that *exchanging the order of two odd operators gives a minus sign.* With this the normal ordering of P^μ reads as

$$:P^\mu: = \sum_s \int_{\mathbb{R}^{d-1}} k^\mu \left(a_s^\dagger(\mathbf{k})\, a_s(\mathbf{k}) + b_s^\dagger(\mathbf{k})\, b_s(\mathbf{k}) \right) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad . \tag{5.86}$$

Acting with this operator on single particle states gives

$$:P^\mu:|k, s, \pm\rangle = k^\mu\, |k, s, \pm\rangle \quad , \tag{5.87}$$

hence the label $k = (\omega_{\mathbf{k}}, \mathbf{k})$ has the physical interpretation of the on-shell momentum of a particle with mass m. Note that both a-type and b-type particles have positive energies and identical mass m. The normal ordered operator associated with the relativistic angular momentum (5.71) is more complicated to work out, hence we shall skip the details. Even without looking into the details, one sees that due to the second term in (5.71), which captures the intrinsic angular momentum a.k.a. spin $S^{\mu\nu}$, the single particle states $|k, s, \pm\rangle$ describe spin $\frac{1}{2}$ particles. The difference between

a-type and b-type particles lies in their electric charge: The operator associated with (5.72) is given in terms of annihilation and creation operators by

$$Q = \sum_s \int_{\mathbb{R}^{d-1}} \Big(a_s^\dagger(\mathbf{k})\, a_s(\mathbf{k}) + b_s(\mathbf{k})\, b_s^\dagger(\mathbf{k}) \Big) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \tag{5.88}$$

which after normal ordering (performed as above with minus signs when exchanging odd operators) becomes

$$:Q: = \sum_s \int_{\mathbb{R}^{d-1}} \Big(a_s^\dagger(\mathbf{k})\, a_s(\mathbf{k}) - b_s^\dagger(\mathbf{k})\, b_s(\mathbf{k}) \Big) \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad . \tag{5.89}$$

Acting on single particle states gives

$$:Q:|k,s,\pm\rangle = \pm\, |k,s,\pm\rangle \quad , \tag{5.90}$$

which means that a-type particles have positive charge and b-type particles have negative charge.

Summing up, we have seen that the quantization of the Dirac field gives two types of fermionic spin $\frac{1}{2}$ particles, which have identical mass m but differ by a sign in their electric charge. Such pairs are typically called particle/antiparticle pairs and a physical example is the electron/positron pair. To avoid any misconceptions, I would like to stress that the existence of antiparticles has nothing to do with the fermionic spin $\frac{1}{2}$ structure of the Dirac field, but rather with the fact that Dirac spinors are complex-valued. As already indicated in the itemization above, when studying a complex Klein-Gordon field, one also finds two independent types of annihilation and creation operators, a-type and b-type, that give rise to particle/antiparticle pairs. The Hermiticity condition $\Phi(x)^\dagger = \Phi(x)$ for the real Klein-Gordon field forces the b-type operators to coincide with the a-type operators (compare with (3.20)). This is why people often say that "the particles associated with a real field are their own antiparticles".

5.4 Feynman propagator for the Dirac field

To perform perturbative QFT calculations in the spirit of Chapter 4 that involve also interacting Dirac quantum fields, we have to understand the Feynman propagator for the Dirac field. From our explicit expressions for the Heisenberg picture field operators (5.81), we immediately observe that the 2-point correlation functions

$$\langle 0|\Psi_\alpha(x)\, \Psi_\beta(y)|0\rangle = 0 \quad , \qquad \langle 0|\overline{\Psi}_\alpha(x)\, \overline{\Psi}_\beta(y)|0\rangle = 0 \tag{5.91}$$

for two Dirac fields or two adjoint Dirac fields are zero. (The indices α, β denote the components of spinors.) The mixed 2-point correlation functions $\langle 0|\Psi_\alpha(x)\, \overline{\Psi}_\beta(y)|0\rangle$ and $\langle 0|\overline{\Psi}_\alpha(x)\, \Psi_\beta(y)|0\rangle$ between Dirac and adjoint Dirac fields are, as we will see, nontrivial. To avoid notational clutter due to using further indices α, β, it makes sense to introduce the matrix-valued correlation functions

$$\langle 0|\Psi(x)\, \overline{\Psi}(y)|0\rangle \in \mathrm{Mat}_{N\times N}(\mathbb{C}) \quad , \qquad \langle 0|\overline{\Psi}(x)\, \Psi(y)|0\rangle \in \mathrm{Mat}_{N\times N}(\mathbb{C}) \quad , \tag{5.92a}$$

whose components are defined by

$$\langle 0|\Psi(x)\, \overline{\Psi}(y)|0\rangle_{\alpha\beta} := \langle 0|\Psi_\alpha(x)\, \overline{\Psi}_\beta(y)|0\rangle \quad , \qquad \langle 0|\overline{\Psi}(x)\, \Psi(y)|0\rangle_{\alpha\beta} := \langle 0|\overline{\Psi}_\beta(x)\, \Psi_\alpha(y)|0\rangle \quad . \tag{5.92b}$$

(Note the flip of indices in the second definition, which is a convention that will be useful later.) Let us compute the first one in detail. Using (5.81) and that the vacuum state is annihilated by all annihilation operators, we compute

$$\begin{aligned}
\langle 0|\Psi(x)\,\overline{\Psi}(y)|0\rangle &= \sum_{s,r}\int_{\mathbb{R}^{2(d-1)}} \frac{e^{\mathrm{i}\,k\,x-\mathrm{i}\,q\,y}}{\sqrt{2\omega_{\mathbf{k}}}\,\sqrt{2\omega_{\mathbf{q}}}}\,\langle 0|a_s(\mathbf{k})\,a_r^\dagger(\mathbf{q})|0\rangle\; u^s(\mathbf{k})\,\overline{u^r(\mathbf{q})}\,\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}}\,\frac{\mathrm{d}\mathbf{q}}{(2\pi)^{d-1}} \\
&= \int_{\mathbb{R}^{d-1}} \frac{e^{\mathrm{i}\,k\,(x-y)}}{2\omega_{\mathbf{k}}}\;\sum_s u^s(\mathbf{k})\,\overline{u^s(\mathbf{k})}\,\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \\
&= \int_{\mathbb{R}^{d-1}} \frac{e^{\mathrm{i}\,k\,(x-y)}}{2\omega_{\mathbf{k}}}\;(-\mathrm{i}\,\not{k}+m)\,\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \\
&= (-\not{\partial}_x+m)\int_{\mathbb{R}^{d-1}} \frac{e^{\mathrm{i}\,k\,(x-y)}}{2\omega_{\mathbf{k}}}\,\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} = (-\not{\partial}_x+m)\,W_2(x,y) \quad ,
\end{aligned} \tag{5.93}$$

where $\not{\partial}_x = \gamma^\mu \frac{\partial}{\partial x^\mu}$ denotes x-derivatives. In the second step we used the anticommutation relations (5.82), in third step the spin sum identity (5.58) and in the last step we have recognized our old friend, the 2-point Wightman function (3.49b) of the Klein-Gordon field. In words, the correlation function $\langle 0|\Psi(x)\,\overline{\Psi}(y)|0\rangle$ for the Dirac field is obtained by applying the "wrong sign Dirac operator" $(-\not{\partial}_x+m)$ on the correlation function $W_2(x,y)$ of the Klein-Gordon field. That's very nice. By a similar calculation, one finds that

$$\langle 0|\overline{\Psi}(x)\,\Psi(y)|0\rangle = -(\not{\partial}_x+m)\,W_2(x,y) = -(-\not{\partial}_y+m)\,W_2(x,y) \quad , \tag{5.94}$$

which is again very nice.

The time-ordered product from (3.54) can be extended to Dirac fields Ψ and their Dirac adjoints $\overline{\Psi}$, with the slight, but by now familiar, modification that there will be a minus sign whenever two odd operators are permuted. For instance, we have

$$\mathsf{T}\big(\Psi_\alpha(x)\overline{\Psi}_\beta(y)\big) = \begin{cases} \Psi_\alpha(x)\,\overline{\Psi}_\beta(y) & , \text{ if } x^0 \geq y^0 \quad , \\ -\overline{\Psi}_\beta(y)\,\Psi_\alpha(x) & , \text{ if } y^0 \geq x^0 \quad . \end{cases}$$

We define the *Feynman propagator for the Dirac field* as the vacuum expectation value

$$\begin{aligned}
\langle 0|\mathsf{T}\big(\Psi(x)\overline{\Psi}(y)\big)|0\rangle &= \begin{cases} \langle 0|\Psi(x)\,\overline{\Psi}(y)|0\rangle & , \text{ if } x^0 \geq y^0 \quad , \\ -\langle 0|\overline{\Psi}(y)\,\Psi(x)|0\rangle & , \text{ if } y^0 \geq x^0 \quad , \end{cases} \\
&= \begin{cases} (-\not{\partial}_x+m)\,W_2(x,y) & , \text{ if } x^0 \geq y^0 \quad , \\ (-\not{\partial}_x+m)\,W_2(y,x) & , \text{ if } y^0 \geq x^0 \quad , \end{cases} \\
&= (-\not{\partial}_x+m)\,\Delta_F(x-y) \quad ,
\end{aligned} \tag{5.95}$$

where $\Delta_F(x-y)$ is the Feynman propagator for the Klein-Gordon field. Using our Fourier integral formula (3.57) for the latter, we obtain the following Fourier integral formula

$$S_F(x-y) := \langle 0|\mathsf{T}\big(\Psi(x)\overline{\Psi}(y)\big)|0\rangle = \lim_{\epsilon\to 0}\int_{\mathbb{R}^d} \frac{-\mathrm{i}\,(-\mathrm{i}\,\not{k}+m)}{k^2+m^2-\mathrm{i}\,\epsilon}\; e^{\mathrm{i}\,k\,(x-y)}\,\frac{\mathrm{d}k}{(2\pi)^d} \tag{5.96}$$

for the Feynman propagator for the Dirac field. Note that this Feynman propagator is a Green's function for the Dirac equation, i.e.

$$\begin{aligned}(\not{\partial}_x + m)\, S_F(x-y) &= (\not{\partial}_x + m)\,(-\not{\partial}_x + m)\,\Delta_F(x-y) \\ &= (-\partial^2 + m^2)\,\Delta_F(x-y) \;=\; -\mathrm{i}\,\delta(x-y) \quad . \end{aligned} \tag{5.97}$$

Taking again care of the signs associated with permuting odd operators, Wick's Theorem 4.1 generalizes to the Dirac field, and so does its variant from Corollary 4.3 that is the key for perturbative QFT calculations. For example, using our shorthand notation from Section 4.2, we find that

$$\langle 0|\mathsf{T}(\Psi_1\,\overline{\Psi}_2\,\Psi_3\,\overline{\Psi}_4)|0\rangle = (S_F)_{12}\,(S_F)_{34} - (S_F)_{14}\,(S_F)_{32} \quad , \tag{5.98}$$

where the minus sign arises from the odd permutation $(1234) \to (1432)$. Comparing this with the case of the Klein-Gordon field from Example 4.4, we note that there is no $(S_F)_{13}\,(S_F)_{24}$ contribution. This follows from our observation above that the correlation functions for the Dirac field are only nontrivial when evaluated among Ψ and $\overline{\Psi}$, i.e. all correlation functions involving only Ψ's or $\overline{\Psi}$'s are zero. From this observation one also deduces that

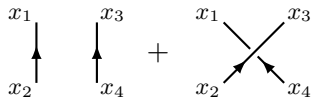
$$\langle 0|\mathsf{T}\Big(\prod_{i=1}^{n}\Psi_i \prod_{j=1}^{m}\overline{\Psi}_j\Big)|0\rangle = 0 \qquad (\text{for } n \neq m) \quad , \tag{5.99}$$

so time-ordered $n+m$-point functions are trivial unless there is an equal number of Dirac Ψ and adjoint Dirac $\overline{\Psi}$ fields.

As a final but important comment, I would like to add that the two points $x, y \in \mathbb{R}^d$ in the Feynman propagator for the Dirac field (5.96) play very different roles as x is the position of the Ψ operator and y the position of the $\overline{\Psi}$ operator. This in particular means that we have to be more careful when drawing Feynman diagrams involving Dirac fields in order to distinguish between the Ψ's and the $\overline{\Psi}$'s. The standard way to deal with this issue is to draw a directed line

$$S_F(x-y) \;=\; \langle 0|\mathsf{T}(\Psi(x)\overline{\Psi}(y))|0\rangle \;=\; x \longleftarrow y \tag{5.100}$$

with arrow pointing from $\overline{\Psi}$ to Ψ. The direction of this arrow is chosen to coincide with the forward-in-time flow of a-type particles; indeed, from (5.81) we see that $\overline{\Psi}$ creates a-type particles and that Ψ annihilates a-type particles. (Alternatively, the arrow depicts the backward-in-time flow of b-type particles.) With these conventions, the time-ordered 4-point function in (5.98) admits the following graphical representation by Feynman diagrams

$\langle 0|\mathsf{T}(\Psi(x_1)\,\overline{\Psi}(x_2)\,\Psi(x_3)\,\overline{\Psi}(x_4))|0\rangle \;=$ 

. (5.101)

It is important to stress that these arrows are a different concept than the momentum flow arrows that we have used in our momentum space Feynman rules in Section 4.6. Hence, when discussing scattering amplitudes involving Dirac fields one has to keep track of both the flow of a-type particles and the momentum flow in Feynman diagrams. We will see later examples of such scattering amplitudes when we study quantum electrodynamics (QED) in Chapter 7.

Further reading

For more details about the Dirac field and its quantization, see e.g. Nastase (Chapters 12-14), Srednicki (Chapter 33 onwards), Greiner/Reinhardt (Chapter 5) and Maggiore (Chapters 3.4 and 4.2) from our reading list in Section 1.3.

Chapter 6

Free quantum electromagnetic field

This chapter studies the canonical quantization of the electromagnetic field and explains how to deal with the gauge symmetry of this system.

6.1 Classical electromagnetism

Let us start by recalling some aspects of classical electromagnetism in $d = 4$ spacetime dimensions before moving to the general case of d dimensions. Maxwell's theory of electromagnetism (in $d = 4$) is traditionally formulated in terms of the electric field $\mathbf{E} = (E_1, E_2, E_3)$ and the magnetic field $\mathbf{B} = (B_1, B_2, B_3)$, whose dynamics is governed by Maxwell's equations

$$\nabla \cdot \mathbf{E} = \rho \quad , \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad , \tag{6.1a}$$

$$\nabla \cdot \mathbf{B} = 0 \quad , \qquad \nabla \times \mathbf{B} = \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} \quad . \tag{6.1b}$$

The charge density ρ and the current density $\mathbf{j} = (j^1, j^2, j^3)$ are required to satisfy the charge conservation law

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad . \tag{6.2}$$

Recalling that every divergence-free vector field is the curl of another vector field, we can solve the homogeneous Maxwell equation $\nabla \cdot \mathbf{B} = 0$ and write without loss of generality

$$\mathbf{B} = \nabla \times \mathbf{A} \quad . \tag{6.3a}$$

Inserting this into the other homogeneous Maxwell equation and recalling that every curl-free vector field is the gradient of a function, we can write

$$\mathbf{E} = \nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \quad . \tag{6.3b}$$

The tuple $(\phi, \mathbf{A}) = (\phi, A_1, A_2, A_3)$ is called the *electromagnetic potential* and it can be used as an alternative set of variables, in contrast to the traditional variables $\mathbf{E}$ and $\mathbf{B}$, to formulate

electromagnetism. Since in these variables the homogeneous Maxwell equations are automatically solved (by construction), we are left with the two inhomogeneous equations

$$\nabla^2\phi - \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) \;=\; \rho \quad , \qquad \frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} + \nabla(\nabla\cdot\mathbf{A}) - \frac{\partial}{\partial t}\nabla\phi \;=\; \mathbf{j} \quad . \tag{6.4}$$

The variables $(\phi, \mathbf{A})$ turn out to be more fundamental than the $(\mathbf{E}, \mathbf{B})$ variables: As we will see later, it is the electromagnetic potential $(\phi, \mathbf{A})$ that couples to a charged Dirac field and not the $(\mathbf{E}, \mathbf{B})$ fields. There are further arguments why the electromagnetic potential is more fundamental and I recommend you to google e.g. the "Aharonov-Bohm effect". However, there is a price to pay: The defining equations (6.3) do *not* fix $(\phi, \mathbf{A})$ uniquely for a given $(\mathbf{E}, \mathbf{B})$, so that one has to deal with redundancies in the description. Indeed, you can easily check that the potentials

$$(\phi, \mathbf{A}) \;\sim\; \Big(\phi + \frac{\partial\alpha}{\partial t}, \mathbf{A} + \nabla\alpha\Big) \tag{6.5}$$

define the same $\mathbf{E}$ and $\mathbf{B}$ fields, for any choice of function $\alpha(x)$ on the Minkowski spacetime. Hence, such electromagnetic potentials should be regarded as being physically equivalent, which can be formalized mathematically by working with an equivalence relation on the set of potentials. We will come back to this issue later in this section.

The equations (6.4) do not look very pleasant and it is also not obvious if they are covariant under Poincaré transformations. These two issues can be solved by introducing the relativistic electromagnetic potential A_μ in terms of the covector field

$$A := \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} := \begin{pmatrix} \phi \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} \tag{6.6}$$

and the relativistic current j^μ in terms of the vector field

$$j := \begin{pmatrix} j^0 \\ j^1 \\ j^2 \\ j^3 \end{pmatrix} := \begin{pmatrix} \rho \\ j^1 \\ j^2 \\ j^3 \end{pmatrix} \tag{6.7}$$

on the Minkowski spacetime $\mathbb{R}^4$. Let us further define the field strength tensor by antisymmetrized partial differentiation

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \tag{6.8}$$

A quick check shows that the matrix represented by $F_{\mu\nu}$ reads as

$$F = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \quad , \tag{6.9}$$

i.e. it packages the $\mathbf{E}$ and $\mathbf{B}$ fields into a single $(0,2)$-tensor field, and that the equation

$$\partial_\mu F^{\mu\nu} = -j^\nu \tag{6.10a}$$

captures the inhomogeneous Maxwell equations. When written in terms of A_μ, the latter equation reads as

$$-\partial^2 A_\mu + \partial_\mu(\partial_\nu A^\nu) = j_\mu \quad , \tag{6.10b}$$

where we recall that $\partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$. The components of this equation are precisely (6.4), hence we found a neat way to write everything in a manifestly Poincaré covariant form.

It is now obvious how to generalize electromagnetism to the d-dimensional Minkowski spacetime $(\mathbb{R}^d, \eta)$: We simply introduce a d-dimensional covector field A_μ for the electromagnetic potential, a d-dimensional vector field j^μ satisfying $\partial_\mu j^\mu = 0$ for the current and demand the equation of motion (6.10) to hold on $\mathbb{R}^d$. A nice feature of this equation of motion is that it is the Euler-Lagrange equation of the following action functional

$$S_{\mathrm{MW}}[A] := \int_{\mathbb{R}^d} \Big(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu j^\mu \Big) \, \mathrm{d}x \quad , \tag{6.11}$$

as one can easily check. For a trivial current $j^\mu = 0$, i.e. for electromagnetism in vacuum, this action is clearly invariant under (active) Poincaré transformations

$$T \,:\, A_\mu(x) \longmapsto (TA)_\mu(x) = \Lambda_\mu{}^\nu \, A_\nu(\Lambda^{-1}(x-b)) \quad , \tag{6.12}$$

hence we get from Theorem 2.6 Noether currents and conserved charges that capture the relativistic momentum and angular momentum of the electromagnetic potential. Due to time constraints, we do not study these charges in detail and simply state the main outcome: Since A_μ is a multicomponent field transforming in the covector representation of the Lorentz group, one finds that this field carries a nontrivial internal angular momentum, a.k.a. spin, which upon quantization will give rise to spin 1 particles, the photons.

The main new feature of electromagnetism, which we didn't have in any of our previous examples (Klein-Gordon, Dirac, ...), is that the action functional (6.11) has a huge (really, really huge!) amount of internal symmetries, given by the transformations

$$T_\alpha \,:\, A_\mu(x) \longmapsto (T_\alpha A)_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x) \tag{6.13}$$

that are labeled by functions $\alpha(x)$ on the Minkowski spacetime $\mathbb{R}^d$. Indeed, these transformations leave invariant the field strength tensor

$$F_{\mu\nu} \longmapsto \partial_\mu(A_\nu + \partial_\nu\alpha) - \partial_\nu(A_\mu + \partial_\mu\alpha) = F_{\mu\nu} + \partial_\mu\partial_\nu\alpha - \partial_\nu\partial_\mu\alpha = F_{\mu\nu} \tag{6.14}$$

and also the action

$$S_{\mathrm{MW}}[T_\alpha A] = S_{\mathrm{MW}}[A] + \int_{\mathbb{R}^d} \partial_\mu\alpha \, j^\mu \, \mathrm{d}x = S_{\mathrm{MW}}[A] - \int_{\mathbb{R}^d} \alpha \, \partial_\mu j^\mu \, \mathrm{d}x = S_{\mathrm{MW}}[A] \quad , \tag{6.15}$$

where in the second step we have integrated by parts and the last step follows from the charge conservation law $\partial_\mu j^\mu = 0$, see (6.2).

Let me emphasize that the main feature here is that α is *not* required to be a constant, as it was the case for the internal symmetries of complex Klein-Gordon or Dirac fields, see e.g. Example 2.5. This means that the transformations (6.13) are *local symmetries* in the sense that they are allowed to act differently at different points $x \in \mathbb{R}^d$. Such local symmetries are also often called *gauge symmetries* in the QFT literature and theories that admit gauge symmetries are called *gauge theories*. Gauge theories, such as electromagnetism and its non-Abelian generalization called Yang-Mills theory, are extremely important in fundamental physics and they are one of the main players in the standard model of particle physics.

As already mentioned above in (6.5), the gauge symmetries (6.13) should *not* be regarded as some kind of physical symmetries of the system (such as translations or Lorentz transformations), but they are rather a consequence of using a description in terms of variables that has some redundancies. This means that one should consider two potentials that are related by a gauge transformation to be physically equivalent, which is mathematically realized by introducing the equivalence relation

$$A_\mu \sim (T_\alpha A)_\mu = A_\mu + \partial_\mu \alpha \quad . \tag{6.16}$$

One way to proceed would thus be to work with the quotient set of all electromagnetic potentials modulo gauge equivalence, but this turns out to be impractical for studying quantization. A more suitable approach is to make use of the gauge symmetries to fix A_μ to be of a particularly nice and useful form; this is called *gauge fixing*. There are different choices of gauge fixings, but the most prominent and (in my personal opinion) most relevant one is given by the Poincaré invariant *Lorenz gauge fixing*

$$\partial_\mu A^\mu = 0 \quad . \tag{6.17}$$

(Note that here is no typo: Lorenz and Lorentz are two different persons.) It is important to note that every A_μ is gauge equivalent to a potential that satisfies the Lorenz gauge fixing condition, hence imposing this gauge fixing does not alter the physical content of the theory. In more detail, given any A_μ, we define α by solving the inhomogeneous wave equation $-\partial^2 \alpha = \partial_\mu A^\mu$. (It is well-known that this differential equation admits solutions.) Then the gauge transformed potential $(T_\alpha A)_\mu = A_\mu + \partial_\mu \alpha$ satisfies the Lorenz gauge fixing condition $\partial_\mu (T_\alpha A)^\mu = \partial_\mu A^\mu + \partial^2 \alpha = 0$. It is important to stress that the Lorenz gauge fixing (6.17) does *not* fix a unique representative of a gauge equivalence class, hence people call it often a partial gauge fixing. In fact, if A_μ satisfies $\partial_\mu A^\mu = 0$, then so does $(T_\alpha A)_\mu = A_\mu + \partial_\mu \alpha$ for every function α that satisfies the homogeneous wave equation $-\partial^2 \alpha = 0$. To obtain a full gauge fixing, one can use this residual gauge symmetry in order to fix the value of, say, the 0-component A_0. (For $j^\mu = 0$, one can fix $A_0 = 0$, called *temporal gauge*, which together with (6.17) also implies the *Coulomb gauge* $\nabla \cdot \mathbf{A} = 0$.) One typically does not make use of such full gauge fixings, because they have the unpleasant feature of being not manifestly covariant under Poincaré transformations, which would considerably complicate QFT computations. However, from the existence of such full gauge fixings, one can draw a useful conclusion: Of the d components of A_μ, for $\mu = 0, 1, \ldots, d-1$, only $d-2$ degrees of freedom are nonredundant because 1 gets fixed by (6.17) and 1 more gets fixed by the additional gauge fixing

for A_0. Hence, we see that the electromagnetic potential (modulo gauge symmetry) describes $d-2$ physical degrees of freedom, which for $d=4$ are the well-known two independent polarizations of light.

The Lorenz gauge fixing condition (6.17) can be implemented by introducing a Lagrange multiplier ξ and modifying the action (6.11) according to

$$S_{\text{MW+gf}}[A,\xi] := \int_{\mathbb{R}^d} \Big(-\frac{1}{4}\, F^{\mu\nu}\, F_{\mu\nu} - \frac{\xi}{2}\, (\partial_\mu A^\mu)^2 + A_\mu\, j^\mu \Big)\, \mathrm{d}x \quad . \tag{6.18}$$

The Euler-Lagrange equation for A_μ then reads as

$$-\partial^2 A_\mu + (1-\xi)\, \partial_\mu(\partial_\nu A^\nu) \;=\; j_\mu \tag{6.19}$$

and the one for ξ enforces the Lorenz gauge fixing

$$\frac{1}{2}(\partial_\mu A^\mu)^2 \;=\; 0 \quad \Longleftrightarrow \quad \partial_\mu A^\mu \;=\; 0 \quad . \tag{6.20}$$

Fixing $\xi = 1$, one obtains yet another description in terms of the action

$$\begin{aligned} \widetilde{S}_{\text{MW}}[A] \;&:=\; \int_{\mathbb{R}^d} \Big(-\frac{1}{4}\, F^{\mu\nu}\, F_{\mu\nu} - \frac{1}{2}\, (\partial_\mu A^\mu)^2 + A_\mu\, j^\mu \Big)\, \mathrm{d}x \\ &=\; \int_{\mathbb{R}^d} \Big(-\frac{1}{2}\, \partial^\mu A^\nu\, \partial_\mu A_\nu + \frac{1}{2}\, \partial^\mu A^\nu\, \partial_\nu A_\mu - \frac{1}{2}\, (\partial_\mu A^\mu)^2 + A_\mu\, j^\mu \Big)\, \mathrm{d}x \\ &=\; \int_{\mathbb{R}^d} \Big(-\frac{1}{2}\, \partial^\mu A^\nu\, \partial_\mu A_\nu + A_\mu\, j^\mu \Big)\, \mathrm{d}x \quad , \end{aligned} \tag{6.21a}$$

where the last step uses twice integration by parts, together with the constraint

$$\partial_\mu A^\mu = 0 \quad . \tag{6.21b}$$

Since it is crucial to not forget this constraint when working with this form of the action, let me write it again in a single colored box

$$\widetilde{S}_{\text{MW}}[A] \;=\; \int_{\mathbb{R}^d} \Big(-\frac{1}{2}\, \partial^\mu A^\nu\, \partial_\mu A_\nu + A_\mu\, j^\mu \Big)\, \mathrm{d}x$$
$$\textbf{subject to the constraint} \quad \partial_\mu A^\mu \;=\; 0 \quad . \tag{6.22}$$

The Euler-Lagrange equation corresponding to this action is simply the wave equation (or massless Klein-Gordon equation)

$$-\partial^2 A_\mu \;=\; 0 \tag{6.23}$$

for each component $\mu = 0, \dots, d-1$. This last description is most suitable for discussing the quantization of the electromagnetic potential A_μ in the next section. Note that (6.22) is *not* invariant under all gauge transformations (6.13), but only under the residual gauge transformations preserving the Lorenz gauge fixing, i.e.

$$T_\alpha \;:\; A_\mu \longmapsto (T_\alpha A)_\mu = A_\mu + \partial_\mu \alpha \qquad \text{with} \quad -\partial^2 \alpha = 0 \quad . \tag{6.24}$$

6.2 Canonical quantization à la Gupta-Bleuler

In what follows we will set the current $j^\mu = 0$ to zero and consider the free electromagnetic potential. In the next chapter we will see how coupling A_μ to a charged Dirac field provides a quantum field theoretic model for a current.

The aim of this section is to discuss the quantization of the electromagnetic potential A_μ, which as we will see in a moment is more complicated than in our previous cases due to the presence of gauge symmetries. Let us first note that ignoring the gauge symmetries in the quantization procedure is *not* an option. Indeed, starting from the usual action (6.11) with $j^\mu = 0$, one computes the canonical momenta

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -F^{0\mu} \quad . \tag{6.25}$$

But wait: $\Pi^0 = -F^{00} = 0$ due to antisymmetry of $F^{\mu\nu} = -F^{\nu\mu}$, so the pair (Π^0, A_0) is not a good pair of canonically conjugate variables. Indeed, it is impossible to demand canonical Poisson bracket relations of the form $\{A_0(\mathbf{x}), \Pi^0(\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y})$, or upon quantization canonical commutation relations of the form $[A_0(\mathbf{x}), \Pi^0(\mathbf{y})] = \mathrm{i}\,\delta(\mathbf{x}-\mathbf{y})$, since $\Pi^0 = 0$ is identically zero. That this issue is indeed related to gauge symmetry can be seen quite easily: As discussed in the previous section, we can use gauge transformations to fix $A_0 = 0$ in temporal gauge, so this variable is not dynamical and hence it doesn't require a canonical momentum. Things get much better when we start from the action (6.18) that enforces Lorenz gauge fixing by a Lagrange multiplier. The canonical momenta for this action are

$$\Pi^\mu = -F^{0\mu} - \xi\, \eta^{0\mu}\, (\partial_\rho A^\rho) \quad , \tag{6.26}$$

so we see that for $\xi \neq 0$ also A_0 gets a nonvanishing canonical momentum Π^0.

One could now plough through the canonical quantization of this system, but we shall rather follow a simpler approach that starts from the equivalent action (6.22) where Lorenz gauge is imposed as a constraint by hand. The canonical momenta for the action (6.22) are very simple and given by

$$\Pi^\mu = \dot{A}^\mu \quad . \tag{6.27}$$

That's nice and looks pretty much like what one gets for a Klein-Gordon field, see Example 2.13. The Hamiltonian associated to the action (6.22) is given by

$$H[\Pi, A] \;=\; \int_{\mathbb{R}^{d-1}} \frac{1}{2} \Big(\Pi^\mu\, \Pi_\mu + \partial^i A^\mu\, \partial_i A_\mu \Big)\, \mathrm{d}\mathbf{x} \quad . \tag{6.28}$$

Superficially, this looks pretty much like the Klein-Gordon Hamiltonian (2.58) with zero mass $m^2 = 0$, but there is one key difference: The potential A_μ is a multicomponent field, with $\mu = 0, 1, \ldots, d-1$. Splitting into the time-like component 0 and the spatial components j, the Hamiltonian reads as

$$H[\Pi, A] \;=\; \int_{\mathbb{R}^{d-1}} \frac{1}{2} \Big(-(\Pi^0)^2 - (\nabla A_0)^2 + \sum_{j=1}^{d-1} \Big((\Pi^j)^2 + (\nabla A_j)^2 \Big) \Big)\, \mathrm{d}\mathbf{x} \quad , \tag{6.29}$$

where we use that $\eta_{00} = \eta^{00} = -1$ in the Minkowski metric. So each (Π^j, A_j) contributes to the Hamiltonian in exactly the same way as a massless Klein-Gordon field (Π, Φ) would do, however (Π^0, A_0) has the wrong sign and contributes negatively to the Hamiltonian. This looks like a disaster! The Hamiltonian/energy of the system is unbounded from below, which is a potential source of unphysical features such as instability of the system, nonexistence of a ground state/vacuum, etc. The good news is that this disaster can be prevented by dealing correctly with the constraint $\partial_\mu A^\mu = 0$ that comes together with the action (6.22) and its residual gauge symmetries (6.24). Indeed, at the classical level, we can use the residual gauge symmetries to fix $\Pi^0 = 0$ and $A_0 = 0$, hence the negative contribution to the Hamiltonian can be avoided. We will see later in this section that an analogous conclusion holds true for the quantized theory.

After all these words of warning, let us now finally carry out the quantization of the electromagnetic potential A_μ and see what this gives. We follow the approach of Gupta and Bleuler, which consists of carrying out the following steps:

1. We start with quantizing the canonical variables (Π^μ, A_μ) and the Hamiltonian (6.28) that is associated with the action in (6.22). The constraint $\partial_\mu A^\mu = 0$ and the residual gauge symmetries (6.24) are ignored at this point.

2. We implement the Lorenz gauge fixing constraint at the level of the QFT and analyze the quantum analog of the residual gauge symmetries.

Carrying out the first step is completely analogous to what we have done for the Klein-Gordon field in Sections 3.1 and 3.3, hence we can be relatively fast at this point. Quantization is achieved by promoting A_μ and Π^μ to operators that satisfy the equal-time commutation relations

$$\big[A_\mu(\mathbf{x}), A_\nu(\mathbf{y})\big] \,=\, 0 \,=\, \big[\Pi^\mu(\mathbf{x}), \Pi^\nu(\mathbf{y})\big] \quad , \qquad \big[A_\mu(\mathbf{x}), \Pi^\nu(\mathbf{y})\big] \,=\, \mathrm{i}\,\delta_\mu^\nu\,\delta(\mathbf{x}-\mathbf{y}) \quad . \tag{6.30}$$

To obtain the Heisenberg picture field operators $A_\mu(x)$ and $\Pi^\mu(x)$, one has to solve Heisenberg's equations

$$\frac{\partial}{\partial t} A_\mu(x) \,=\, \mathrm{i}\,\big[H, A_\mu(x)\big] \quad , \qquad \frac{\partial}{\partial t} \Pi^\mu(x) \,=\, \mathrm{i}\,\big[H, \Pi^\mu(x)\big] \quad , \tag{6.31}$$

subject to the initial conditions $A_\mu(0,\mathbf{x}) = A_\mu(\mathbf{x})$ and $\Pi^\mu(0,\mathbf{x}) = \Pi^\mu(\mathbf{x})$, where H denotes the Hamiltonian operator that is obtained by quantizing (6.28). (Normal ordering of the Hamiltonian operator H is inessential at this point, because under the commutator we have that $[:H:, -] = [H, -]$.) Introducing as in Section 3.1 annihilation and creation operators, one finds

$$A_\mu(x) \,=\, \int_{\mathbb{R}^{d-1}} \frac{1}{\sqrt{2\,|\mathbf{k}|}} \sum_{\lambda=0}^{d-1} \Big(a_\lambda(\mathbf{k})\,\epsilon_\mu^\lambda(\mathbf{k})\,e^{\mathrm{i}\,k\,x} + a_\lambda^\dagger(\mathbf{k})\,\epsilon_\mu^\lambda(\mathbf{k})\,e^{-\mathrm{i}\,k\,x} \Big)\, \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \tag{6.32}$$

and that $\Pi^\mu(x) = \dot{A}^\mu(x)$ is given by a time derivative, hence it contains no new information. A few remarks are in order:

- The relativistic momentum $k = (|\mathbf{k}|, \mathbf{k}) \in \mathbb{R}^d$ satisfies the *massless* energy-momentum relation (or on-shell condition)

$$k^2 = \eta_{\mu\nu}\, k^\mu\, k^\nu = 0 \tag{6.33}$$

because there is no mass term in the Hamiltonian (6.28) (as well as in the action (6.22) and its Euler-Lagrange equation (6.23)). For comparison with the formulas from Section 3.1, use that the energy $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$ is given in the massless case by the Euclidean norm $|\mathbf{k}|$ of the spatial momentum.

- There is a sum over a basis of polarization covectors $\{\epsilon^\lambda(\mathbf{k}) \in \mathbb{R}^d : \lambda = 0, \dots, d-1\}$, which arises because A_μ is a multicomponent field. (We have seen a similar sum over spinor polarizations in the description of the Dirac quantum field (5.81).) A possible choice of basis would be $\epsilon^\lambda_\mu(\mathbf{k}) = \delta^\lambda_\mu$, but this choice isn't the best for our discussion of the Hilbert space below. A better choice is given by taking a basis of polarization covectors that is adapted to time-like, transversal and longitudinal polarizations. In more detail, splitting covectors into their 0 component and spatial components, we choose

$$\epsilon^0(\mathbf{k}) = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} \in \mathbb{R}^d \quad , \quad \epsilon^j(\mathbf{k}) = \begin{pmatrix} 0 \\ \boldsymbol{\epsilon}^j(\mathbf{k}) \end{pmatrix} \in \mathbb{R}^d \quad , \tag{6.34a}$$

such that

$$\boldsymbol{\epsilon}^j(\mathbf{k}) \cdot \mathbf{k} = 0 \qquad \text{for } j = 1, \dots, d-2 \quad , \tag{6.34b}$$

$$\boldsymbol{\epsilon}^{d-1}(\mathbf{k}) = \frac{\mathbf{k}}{|\mathbf{k}|} \quad . \tag{6.34c}$$

This means that $\epsilon^0(\mathbf{k}) \in \mathbb{R}^d$ is a time-like polarization, $\epsilon^1(\mathbf{k}), \dots, \epsilon^{d-2}(\mathbf{k}) \in \mathbb{R}^d$ are transversal spatial polarizations and $\epsilon^{d-1}(\mathbf{k}) \in \mathbb{R}^d$ is a longitudinal spatial polarization. We choose these polarization covectors to be orthonormal with respect to the inverse Minkowski metric, i.e.

$$\epsilon^\lambda_\mu(\mathbf{k})\, \epsilon^{\lambda'}_\nu(\mathbf{k})\, \eta^{\mu\nu} = \eta^{\lambda\lambda'} \quad , \tag{6.35a}$$

from which the following polarization sum identity follows

$$\sum_{\lambda=0}^{d-1} \eta_{\lambda\lambda}\, \epsilon^\lambda_\mu(\mathbf{k})\, \epsilon^\lambda_\nu(\mathbf{k}) = \eta_{\mu\nu} \quad . \tag{6.35b}$$

(These are the covector analogs of the spinor formulas (5.57) and (5.58) from our discussion of the Dirac field.)

- The annihilation and creation operators carry a polarization label $\lambda = 0, 1, \dots, d-1$ and they satisfy the following commutation relations

$$[a_\lambda(\mathbf{k}), a_{\lambda'}(\mathbf{q})] = 0 = [a^\dagger_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{q})] \quad , \qquad [a_\lambda(\mathbf{k}), a^\dagger_{\lambda'}(\mathbf{q})] = \eta_{\lambda\lambda'}\, (2\pi)^{d-1}\, \delta(\mathbf{k} - \mathbf{q}) \tag{6.36}$$

involving the Minkowski metric $\eta_{\lambda\lambda'}$. Note that the annihilation and creation operators associated with the time-like polarization $\lambda = 0$ have the "wrong sign", which will cause some issues later that however can be resolved by taking into account the constraint $\partial_\mu A^\mu = 0$ and the residual gauge symmetries (6.24).

To build a Hilbert space $\mathcal{H}$, we proceed in complete analogy to our previous examples and introduce a vacuum state $|0\rangle \in \mathcal{H}$ that is characterized by the property of being normalized $\langle 0|0\rangle = 1$ and being annihilated by all annihilation operators

$$a_\lambda(\mathbf{k})|0\rangle \;=\; 0 \quad . \tag{6.37}$$

Multiparticle states are then obtained by acting on $|0\rangle$ with suitably normalized creation operators, leading to

$$|(k_1,\lambda_1),\dots,(k_n,\lambda_n)\rangle \;:=\; \sqrt{2\,|\mathbf{k}_1|}\cdots\sqrt{2\,|\mathbf{k}_n|}\,a^\dagger_{\lambda_1}(\mathbf{k}_1)\cdots a^\dagger_{\lambda_n}(\mathbf{k}_n)|0\rangle \quad . \tag{6.38}$$

Each particle is labeled by a pair (k,λ) consisting of a relativistic momentum $k = (|\mathbf{k}|,\mathbf{k}) \in \mathbb{R}^d$ that satisfies the massless on-shell condition $k^2 = 0$ and a covector polarization index $\lambda = 0, 1, \dots, d-1$. This looks so far quite good and it seems to indicate that the quantum particles associated with the electromagnetic potential A_μ, which are often called *photons*, are massless particles with nontrivial spin. Furthermore, from the commutation relations (6.36), we see that the multiparticle states are symmetric under the exchange of any of its labels, hence our photons are bosons as they should be. After a more careful look however, we recognize the following huge issue:

!!! The states corresponding to photons with a time-like polarization $\lambda = 0$ have negative norm, as one can see by the following calculation

$$\begin{aligned}
\langle q,\lambda'|k,\lambda\rangle \;&=\; \sqrt{2\,|\mathbf{q}|}\,\sqrt{2\,|\mathbf{k}|}\,\langle 0|a_{\lambda'}(\mathbf{q})\,a^\dagger_\lambda(\mathbf{k})|0\rangle \\
&=\; \sqrt{2\,|\mathbf{q}|}\,\sqrt{2\,|\mathbf{k}|}\,\langle 0|\big[a_{\lambda'}(\mathbf{q}),a^\dagger_\lambda(\mathbf{k})\big]|0\rangle \\
&=\; \eta_{\lambda'\lambda}\,2\,|\mathbf{k}|\,(2\pi)^{d-1}\,\delta(\mathbf{q}-\mathbf{k}) \quad .
\end{aligned} \tag{6.39}$$

This issue arises from the "wrong sign" commutation relation for time-like polarizations in (6.36). Negative norm states are an absolute no-go in quantum theory, because the norm of a state is interpreted as a probability which can never be negative.

The good news is that such negative norm states are just an artifact of the gauge symmetry of electromagnetism and that they can be removed by taking into account the Lorenz gauge fixing constraint and its residual gauge symmetries. This brings us to item 2. in the itemization above. Let us start thinking about how to implement the constraint $\partial_\mu A^\mu = 0$ at the level of the QFT. A first attempt would be to demand $\partial_\mu A^\mu = 0$ as an identity for operators. But that's too naive because, as a consequence of (6.32) and the nontrivial commutation relations (6.36), one finds a nontrivial commutator $[\partial_\mu A^\mu(x), A_\nu(y)] \neq 0$ which is incompatible with implementing $\partial_\mu A^\mu = 0$. A smarter way is to implement the constraint at the level of the Hilbert space $\mathcal{H}$ by selecting a subclass of states $|\psi_{\text{phys}}\rangle$ one calls the *physical states*. This idea is called the *Gupta-Bleuler method*,

named after the two physicists Gupta and Bleuler. The operator that selects the physical states is inspired by the Lorenz gauge fixing and it is constructed as follows: Let us denote by

$$A^{(+)}_\mu(x) \;=\; \int_{\mathbb{R}^{d-1}} \frac{1}{\sqrt{2\,|\mathbf{k}|}} \sum_{\lambda=0}^{d-1} a_\lambda(\mathbf{k})\, \epsilon^\lambda_\mu(\mathbf{k})\, e^{\mathrm{i}\, k\, x}\; \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \tag{6.40}$$

the annihilating part of the Heisenberg picture operator (6.32) and consider the operator

$$\begin{aligned} \partial_\mu A^{(+)\mu}(x) \;=\; \partial^\mu A^{(+)}_\mu(x) \;&=\; \int_{\mathbb{R}^{d-1}} \frac{1}{\sqrt{2\,|\mathbf{k}|}} \sum_{\lambda=0}^{d-1} a_\lambda(\mathbf{k})\, \mathrm{i}\, k^\mu\, \epsilon^\lambda_\mu(\mathbf{k})\, e^{\mathrm{i}\, k\, x}\; \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \\ &=\; \mathrm{i} \int_{\mathbb{R}^{d-1}} \sqrt{\frac{|\mathbf{k}|}{2}}\, \Big(a_0(\mathbf{k}) + a_{d-1}(\mathbf{k})\Big)\, e^{\mathrm{i}\, k\, x}\; \frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \end{aligned} \tag{6.41}$$

where in the last equality we have used our particular choice of basis for the polarization covectors in (6.34). The Gupta-Bleuler condition that selects physical states is then given by

$$\partial_\mu A^{(+)\mu}(x)|\psi_{\text{phys}}\rangle \;=\; 0 \quad . \tag{6.42}$$

The collection of all physical states defines a vector subspace $\mathcal{H}_{\text{phys}} \subset \mathcal{H}$ that is called the *physical Hilbert space*. As a consequence of the Gupta-Bleuler condition (6.42), we find that all matrix elements

$$\langle \phi_{\text{phys}}|\partial_\mu A^\mu(x)|\psi_{\text{phys}}\rangle \;=\; \langle \phi_{\text{phys}}|\partial_\mu A^{(+)\mu}(x)\, \psi_{\text{phys}}\rangle + \langle \partial_\mu A^{(+)\mu}(x)\, \phi_{\text{phys}}|\psi_{\text{phys}}\rangle \;=\; 0 \tag{6.43}$$

of the operator $\partial_\mu A^\mu$ between physical states vanish. Hence, the constraint gets implemented not as an operator identity $\partial_\mu A^\mu = 0$, which as we have seen above would be inconsistent, but in a weaker fashion at the level of matrix elements between physical states.

Note that the vacuum state $|0\rangle$ satisfies the Gupta-Bleuler condition $\partial_\mu A^{(+)\mu}(x)|0\rangle = 0$, i.e. it is an element of the physical Hilbert space $\mathcal{H}_{\text{phys}}$, because (6.41) consists only of annihilation operators that annihilate the vacuum. So the zeroth order sanity check is passed. It is instructive to characterize also the physical 1-particle states. Since (6.41) only contains the annihilation operators for the time-like polarization $\lambda = 0$ and the longitudinal polarization $\lambda = d-1$, we find that all transversally polarized states are physical, i.e.

$$\partial_\mu A^{(+)\mu}(x)|k,j\rangle \;=\; 0 \qquad \text{for all } j = 1,\dots,d-2 \quad . \tag{6.44}$$

This is nice as it matches the physical expectation that the polarization of photons is transversal. On the other hand, the time-like polarized state $|k,0\rangle$ and the longitudinal one $|k,d-1\rangle$ do not satisfy the Gupta-Bleuler condition, but one checks that the superposition

$$|k,\text{sp}\rangle \;:=\; |k,0\rangle + |k,d-1\rangle \tag{6.45a}$$

does satisfy this condition

$$\partial_\mu A^{(+)\mu}(x)|k,\text{sp}\rangle \;=\; 0 \quad . \tag{6.45b}$$

The label "sp" refers to spurious, which will be explained below. Summing up, we see that out of the d polarization degrees of freedom (per fixed momentum k) only $d-1$ satisfy the Gupta-Bleuler condition for physical states. These are the $d-2$ transversal polarizations (6.44) and 1 particular superposition (6.45) of the time-like and longitudinal polarizations.

To understand why I called the states (6.45) spurious, let's look at their properties. These states are not only orthogonal to all transversally polarized states, i.e.

$$\langle q, j|k, \mathrm{sp}\rangle \;=\; 0 \qquad \text{for all } j = 1, \dots, d-2 \quad , \tag{6.46}$$

but they are also orthogonal to themselves

$$\begin{aligned} \langle q, \mathrm{sp}|k, \mathrm{sp}\rangle \;&=\; \sqrt{2\,|\mathbf{q}|}\,\sqrt{2\,|\mathbf{k}|}\;\langle 0|(a_0(\mathbf{q}) + a_{d-1}(\mathbf{q}))\,(a_0^\dagger(\mathbf{k}) + a_{d-1}^\dagger(\mathbf{k}))|0\rangle \\ &=\; 2\,|\mathbf{k}|\;\big(\eta_{00} + \eta_{(d-1)(d-1)}\big)\,(2\pi)^{d-1}\,\delta(\mathbf{q}-\mathbf{k}) \;=\; 0 \quad . \end{aligned} \tag{6.47}$$

This is again a consequence of the "wrong sign" commutation relations for time-like polarizations. So the spurious states do not interfere with all other physical states, and not even with themselves. Furthermore, one can show that the spurious states do not contribute to the expectation value of any physical quantity, such as energy, momentum, spin, etc. Let us show this for the energy/Hamiltonian: The Hamiltonian operator obtained from the normal ordered quantization of (6.28) reads as

$$:H: \;=\; \int_{\mathbb{R}^{d-1}} |\mathbf{k}| \;\sum_{\lambda=0}^{d-1} \eta_{\lambda\lambda}\, a_\lambda^\dagger(\mathbf{k})\, a_\lambda(\mathbf{k})\;\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}} \quad , \tag{6.48}$$

from which one checks that

$$\langle q, \mathrm{sp}|:H:|k, \mathrm{sp}\rangle \;=\; 0 \quad . \tag{6.49}$$

All of this implies that the spurious states do not contribute to physics, which means that we can take a quotient of the Hilbert space of physical states $\mathcal{H}_{\mathrm{phys}}$ by identifying all spurious states with 0, i.e.

$$|\psi_{\mathrm{phys}}\rangle \;\sim\; |\psi_{\mathrm{phys}}\rangle + |\mathrm{spurious}\rangle \quad . \tag{6.50}$$

This equivalence relation can be understood as the quantum analog of the residual gauge symmetries (6.24), see e.g. Exercise 7.2 in the textbook by Greiner/Reinhardt. At the level of the 1-particles states, we are then left with $d-2$ degrees of freedom (per fixed momentum k), which are described by the transversally polarized photon states $|k, j\rangle$, for $j = 1, \dots, d-2$.

6.3 Feynman propagator for the photon

The Feynman propagator for the electromagnetic potential/photon is easy to derive by following the same steps as for the Klein-Gordon field in Section 3.4. Let us first note that the 2-point

correlation function can be obtained using (6.32) and a short calculation

$$\begin{aligned}\langle 0|A_\mu(x)\,A_\nu(y)|0\rangle &= \sum_{\lambda,\lambda'=0}^{d-1}\int_{\mathbb{R}^{2(d-1)}}\frac{e^{\mathrm{i}\,k\,x-\mathrm{i}\,q\,y}}{\sqrt{2\,|\mathbf{k}|}\,\sqrt{2\,|\mathbf{q}|}}\,\langle 0|a_\lambda(\mathbf{k})\,a^\dagger_{\lambda'}(\mathbf{q})|0\rangle\,\epsilon^\lambda_\mu(\mathbf{k})\,\epsilon^{\lambda'}_\nu(\mathbf{q})\,\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}}\,\frac{\mathrm{d}\mathbf{q}}{(2\pi)^{d-1}}\\ &= \int_{\mathbb{R}^{d-1}}\frac{e^{\mathrm{i}\,k\,(x-y)}}{2\,|\mathbf{k}|}\,\sum_{\lambda=0}^{d-1}\eta_{\lambda\lambda}\,\epsilon^\lambda_\mu(\mathbf{k})\,\epsilon^\lambda_\nu(\mathbf{k})\,\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}}\\ &= \eta_{\mu\nu}\int_{\mathbb{R}^{d-1}}\frac{e^{\mathrm{i}\,k\,(x-y)}}{2\,|\mathbf{k}|}\,\frac{\mathrm{d}\mathbf{k}}{(2\pi)^{d-1}}\quad .\end{aligned}\tag{6.51}$$

The second step uses the commutation relations (6.36) and the third step follows from the polarization sum identity (6.35b). Note that the scalar integral in the last line is the massless case $m^2=0$ of the 2-point Wightman function (3.49b) for the Klein-Gordon field. This means that we can simply write

$$\langle 0|A_\mu(x)\,A_\nu(y)|0\rangle \;=\; \eta_{\mu\nu}\,W_2(x,y)\big|_{m^2=0}\quad .\tag{6.52}$$

The *Feynman propagator for the photon* is defined as the time-ordered 2-point function

$$\begin{aligned}\langle 0|\mathsf{T}(A_\mu(x)\,A_\nu(y))|0\rangle &= \begin{cases}\langle 0|A_\mu(x)\,A_\nu(y)|0\rangle & ,\ \text{if } x^0\geq y^0\quad ,\\ \langle 0|A_\nu(y)\,A_\mu(x)|0\rangle & ,\ \text{if } y^0\geq x^0\quad ,\end{cases}\\ &= \begin{cases}\eta_{\mu\nu}\,W_2(x,y)\big|_{m^2=0} & ,\ \text{if } x^0\geq y^0\quad ,\\ \eta_{\nu\mu}\,W_2(y,x)\big|_{m^2=0} & ,\ \text{if } y^0\geq x^0\quad ,\end{cases}\\ &= \eta_{\mu\nu}\;\Delta_F(x-y)\big|_{m^2=0}\quad ,\end{aligned}\tag{6.53}$$

where $\Delta_F(x-y)\big|_{m^2=0}$ is the massless case of the Feynman propagator for the Klein-Gordon field. Recalling the Fourier integral formula (3.57) for the latter, we obtain the following useful integral formula

$$(D_F)_{\mu\nu}(x-y)\;:=\;\langle 0|\mathsf{T}(A_\mu(x)\,A_\nu(y))|0\rangle = \lim_{\epsilon\to 0}\int_{\mathbb{R}^d}\frac{-\mathrm{i}\,\eta_{\mu\nu}}{k^2-\mathrm{i}\,\epsilon}\,e^{\mathrm{i}\,k\,(x-y)}\,\frac{\mathrm{d}k}{(2\pi)^d}\quad .\tag{6.54}$$

The photon Feynman propagator is typically visualized in terms of a wiggly line

$$(D_F)_{\mu\nu}(x-y)\;=\;\langle 0|\mathsf{T}(A_\mu(x)\,A_\nu(y))|0\rangle\;=\;x\;\sim\!\sim\!\sim\;y\tag{6.55}$$

in order to distinguish it from the Klein-Gordon and Dirac Feynman propagators that are visualized by a straight line or, respectively, a straight line with an arrow. (See Sections 4.3 and 5.4.)

Further reading

For more details about the electromagnetic field and its quantization, see e.g. Nastase (Chapter 16), Srednicki (Chapters 54 and 55), Greiner/Reinhardt (Chapters 6 and 7) and Maggiore (Chapters 3.5 and 4.3) from our reading list in Section 1.3.

Chapter 7

Quantum electrodynamics (QED)

This chapter studies the coupling between the electromagnetic potential and the Dirac field, leading to quantum electrodynamics (QED). Some simple examples of scattering amplitudes in QED are discussed.

7.1 Minimal gauge coupling

We would like to build a QFT whose field content is given by a Dirac field $\Psi(x)$ and the electromagnetic potential $A_\mu(x)$, and whose dynamics includes a suitable interaction between these fields. The physics we wish to capture by this QFT is that of the interactions between electrons/positrons, which arise as the quantum particles/antiparticles of Ψ, and photons, which arise as the quantum particles of A_μ. The essential ingredient that is needed for defining such a QFT is an action functional $S_{\text{QED}}[\Psi, \overline{\Psi}, A]$ on the whole collection of fields Ψ, $\overline{\Psi}$ and A. Taking for the quadratic parts (i.e. the free actions) the Dirac and Maxwell actions from (5.39) and (6.11), the problem reduces to finding a suitable interaction term S_{int} for

$$\begin{aligned} S_{\text{QED}}[\Psi, \overline{\Psi}, A] \;&=\; S_{\text{Dirac}}[\Psi, \overline{\Psi}] + S_{\text{MW}}[A] + S_{\text{int}}[\Psi, \overline{\Psi}, A] \\ &=\; \int_{\mathbb{R}^d} -\Big(\overline{\Psi}\,(\not{\partial} + m)\,\Psi + \frac{1}{4}\,F^{\mu\nu}\,F_{\mu\nu} \Big)\,\mathrm{d}x \;+\; S_{\text{int}}[\Psi, \overline{\Psi}, A] \quad . \end{aligned} \tag{7.1}$$

Due to the presence of gauge symmetries (6.13) for the electromagnetic potential, which as we have seen are crucial for removing the redundant/unphysical degrees of freedom of A_μ, any consistent choice of interaction term must be gauge invariant. (Otherwise, the interactions would re-introduce unphysical degrees of freedom.) While there are still infinitely many possibilities to define a gauge invariant S_{int}, there is one particularly nice choice that is not only mathematically very pleasing, as it is obtained from a symmetry principle, but also physically realized in nature. This choice is obtained via a construction that is called *gauging* and *minimal coupling*, which we will now describe in some detail.

To motivate the gauging construction, let us recall that the Dirac action S_{Dirac} is invariant under *global* phase rotations

$$(T_\alpha \Psi)(x) \;=\; e^{-\mathrm{i}\,q\,\alpha}\,\Psi(x) \quad , \quad (T_\alpha \overline{\Psi})(x) \;=\; e^{\mathrm{i}\,q\,\alpha}\,\overline{\Psi}(x) \quad , \tag{7.2}$$

for $\alpha \in \mathbb{R}$ a constant. We have introduced here also a parameter $q \in \mathbb{R}$ that has the physical interpretation of the electric charge of the quantum particles associated with Ψ. Indeed, in the presence of q, the equation for charge eigenstates (5.90) reads as $:Q:|k,s,\pm\rangle = \pm q\,|k,s,\pm\rangle$, i.e. the particle/antiparticle has charge $\pm q$. (For electrons/positrons, one would choose $q = -e$, where e is the elementary charge.) On the other hand, the Maxwell action S_{MW} is invariant under the *local* gauge transformations

$$(T_\alpha A)_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x) \quad , \tag{7.3}$$

for $\alpha(x)$ a function on the Minkowski spacetime $\mathbb{R}^d$. While the type of parameter (α is a scalar) is the same in these transformation laws, there is the huge difference that (7.2) is a global (i.e. x-independent) transformation and (7.3) is a much more general local (i.e. x-dependent) transformation. So why don't we try to extend (7.2) to a local transformation

$$(T_\alpha \Psi)(x) \;=\; e^{-\mathrm{i}\, q\, \alpha(x)}\, \Psi(x) \quad , \quad (T_\alpha \overline{\Psi})(x) \;=\; e^{\mathrm{i}\, q\, \alpha(x)}\, \overline{\Psi}(x) \quad , \tag{7.4}$$

which in the literature is often described as *gauging* the global symmetry. Let us recall that we haven't done this previously because the kinetic term in the Dirac action S_{Dirac} isn't invariant under such local transformations, which is a consequence of

$$\partial_\mu (T_\alpha \Psi)(x) = \partial_\mu \Big(e^{-\mathrm{i}\, q\, \alpha(x)}\, \Psi(x) \Big) = e^{-\mathrm{i}\, q\, \alpha(x)} \Big(\partial_\mu \Psi(x) - \mathrm{i}\, q\, \partial_\mu \alpha(x)\, \Psi(x) \Big) \quad . \tag{7.5}$$

But wait: Since we now have learned about the electromagnetic potential, we recognize that the "bad" second term looks suspiciously similar to the transformation law (7.3) of A_μ under gauge transformations. It can be compensated by introducing the *(gauge) covariant derivative*

$$\boxed{D_\mu \Psi \;:=\; \partial_\mu \Psi + \mathrm{i}\, q\, A_\mu \Psi} \tag{7.6}$$

that combines the partial derivative with the electromagnetic potential. Indeed, under a combined local gauge transformation

$$(T_\alpha \Psi)(x) \;=\; e^{-\mathrm{i}\, q\, \alpha(x)}\, \Psi(x) \quad , \tag{7.7a}$$

$$(T_\alpha \overline{\Psi})(x) \;=\; e^{\mathrm{i}\, q\, \alpha(x)}\, \overline{\Psi}(x) \quad , \tag{7.7b}$$

$$(T_\alpha A)_\mu(x) \;=\; A_\mu(x) + \partial_\mu \alpha(x) \tag{7.7c}$$

on *all the fields of the theory*, the covariant derivative transforms very nicely as

$$T_\alpha \;:\; D_\mu \Psi(x) \;\longmapsto\; e^{-\mathrm{i}\, q\, \alpha(x)}\, D_\mu \Psi(x) \quad . \tag{7.8}$$

As a side-remark, note that the gauge covariant derivative (7.6) has a similar structure as the covariant derivative of tensor fields in general relativity. From a geometric perspective, both of these covariant derivatives can be understood as connections on certain types of fiber bundles.

Using the concept of gauge covariant derivatives, there is an extremely simple method, called *minimal coupling*, to promote the free Dirac action S_{Dirac} to an action that is invariant under the combined local gauge transformations (7.7): We simply replace the partial derivative ∂_μ by the covariant derivative D_μ. Together with the free Maxwell action S_{MW}, which we recall is already gauge invariant and hence doesn't need to be altered, we obtain the action

$$S_{\mathrm{QED}}[\Psi,\overline{\Psi},A] \;=\; \int_{\mathbb{R}^d} -\Big(\overline{\Psi}\,(\not{D}+m)\,\Psi + \frac{1}{4}\,F^{\mu\nu}\,F_{\mu\nu}\Big)\,\mathrm{d}x \tag{7.9}$$

that is clearly invariant under the gauge transformations (7.7). Writing $\not{D} = \not{\partial} + \mathrm{i}\,q\,\not{A}$ and comparing with (7.1), we observe that minimal coupling leads to the interaction term

$$S_{\mathrm{int}}[\Psi,\overline{\Psi},A] \;=\; -\mathrm{i}\,q\int_{\mathbb{R}^d}\overline{\Psi}\not{A}\Psi\,\mathrm{d}x \;=\; -\mathrm{i}\,q\int_{\mathbb{R}^d} A_\mu\,\overline{\Psi}\gamma^\mu\Psi\,\mathrm{d}x \quad . \tag{7.10}$$

Comparing this further with (6.11), we note that A_μ couples to the current $j^\mu = -\mathrm{i}\,q\,\overline{\Psi}\gamma^\mu\Psi$ that is build out of the Dirac field and its adjoint.

It is somewhat surprising that this simple gauging and minimal coupling construction, leading to the action functional (7.9), describes the physics of quantum electrodynamics extremely well. What is even more surprising is that *all interactions in the standard model of particle physics* (electromagnetic, weak and strong) are modeled by gauging and minimal coupling a certain global symmetry group, namely the $\mathsf{U}(1)\times\mathsf{SU}(2)\times\mathsf{SU}(3)$ group. We will discuss this in more detail in Chapter 9.

Remark 7.1. As a side-remark, I would like to add that the gauging and minimal coupling construction also works for other types of complex fields, such as e.g. a complex scalar field $\Phi(x)\in\mathbb{C}$. Indeed, gauging the global $\mathsf{U}(1)$ symmetry from Example 2.5 to the local gauge symmetry

$$(T_\alpha\Phi)(x) \;=\; e^{-\mathrm{i}\,q\,\alpha(x)}\,\Phi(x) \quad , \tag{7.11a}$$
$$(T_\alpha\Phi^*)(x) \;=\; e^{\mathrm{i}\,q\,\alpha(x)}\,\Phi^*(x) \quad , \tag{7.11b}$$
$$(T_\alpha A)_\mu(x) \;=\; A_\mu(x)+\partial_\mu\alpha(x) \quad , \tag{7.11c}$$

we can introduce the gauge covariant derivatives

$$D_\mu\Phi \;:=\; \partial_\mu\Phi + \mathrm{i}\,q\,A_\mu\Phi \quad , \quad D_\mu\Phi^* \;:=\; \partial_\mu\Phi^* - \mathrm{i}\,q\,A_\mu\Phi^* \tag{7.12}$$

and the gauge invariant action functional

$$S_{\mathrm{scalarQED}}[\Phi,\Phi^*,A] \;=\; \int_{\mathbb{R}^d} -\Big(D^\mu\Phi^*\,D_\mu\Phi + m^2\,\Phi^*\,\Phi + V(\Phi^*\,\Phi) + \frac{1}{4}\,F^{\mu\nu}\,F_{\mu\nu}\Big)\,\mathrm{d}x \quad . \tag{7.13}$$

This theory is called *scalar (quantum) electrodynamics* with potential V and it can be used as a nice toy-model to investigate an Abelian variant of the Higgs mechanism.

7.2 Feynman rules and scattering amplitudes in QED

Using perturbative quantization techniques similar to those that we have developed for scalar fields in Chapter 4, one can determine the interacting time-ordered n-point functions and scattering amplitudes in quantum electrodynamics. We will skip redoing many of the tedious details and move quickly to a simpler and more intuitive description in terms of Feynman diagrams.

In order to have a well-defined Feynman propagator for the photon field, we have to take into account the gauge symmetries (7.7) and consider as in Chapter 6 a gauge fixed version of the action functional (7.9). Working in Lorenz gauge, this action is given by

$$\begin{aligned}\widetilde{S}_{\mathrm{QED}}[\Psi,\overline{\Psi},A] \;&=\; \int_{\mathbb{R}^d} -\Big(\overline{\Psi}\,(\not{D}+m)\,\Psi + \frac{1}{4}\,F^{\mu\nu}\,F_{\mu\nu} + \frac{1}{2}\,(\partial_\mu A^\mu)^2\Big)\,\mathrm{d}x \\ &=\; \int_{\mathbb{R}^d} -\Big(\overline{\Psi}\,(\not{\partial}+m)\,\Psi + \frac{1}{2}\,\partial^\mu A^\nu\,\partial_\mu A_\nu\Big)\,\mathrm{d}x \;-\; \mathrm{i}\,q\int_{\mathbb{R}^d}\overline{\Psi}\not{A}\Psi\,\mathrm{d}x \quad , \qquad (7.14)\end{aligned}$$

together with the constraint $\partial_\mu A^\mu = 0$. Note that the first term describes the free Dirac and gauge fixed Maxwell actions, and the second term describes a cubic interaction.

Time-ordered n-point functions:

Let us recall from Section 5.4 that the free Dirac Feynman propagator is given by

$$S_F(x-y) \;=\; \lim_{\epsilon\to 0}\int_{\mathbb{R}^d}\frac{-\mathrm{i}\,(-\mathrm{i}\,\not{k}+m)}{k^2+m^2-\mathrm{i}\,\epsilon}\;e^{\mathrm{i}\,k\,(x-y)}\,\frac{\mathrm{d}k}{(2\pi)^d} \;=\; x$$

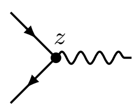
$$y \quad , \qquad (7.15)$$

where the arrow points along the flow of particles (or alternatively against the flow of antiparticles). The free photon Feynman propagator was derived in Section 6.3 and it is given by

$$(D_F)_{\mu\nu}(x-y) \;=\; \lim_{\epsilon\to 0}\int_{\mathbb{R}^d}\frac{-\mathrm{i}\,\eta_{\mu\nu}}{k^2-\mathrm{i}\,\epsilon}\;e^{\mathrm{i}\,k\,(x-y)}\,\frac{\mathrm{d}k}{(2\pi)^d} \;=\; x$$

$$y \quad . \qquad (7.16)$$

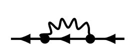
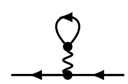
The interaction vertices are generated through the exponential $e^{\mathrm{i}\,S_{\mathrm{int}}}$ in the Gell-Mann and Low formula (4.16). Since the interaction term $S_{\mathrm{int}} = -\mathrm{i}\,q\int_{\mathbb{R}^d}\overline{\Psi}\not{A}\Psi\,\mathrm{d}x$ in QED is cubic, this yields a 3-valent vertex of the form



$$=\; q\,\gamma^\mu\int_{\mathbb{R}^d}\mathrm{d}z \quad . \qquad (7.17)$$

Based on these building blocks, we can now draw and evaluate Feynman diagrams for time-ordered n-point functions in QED. Let us illustrate this by studying an example: The interacting Dirac Feynman propagator to order q^3 in the coupling constant q. The relevant Feynman diagrams are

$$\langle\Omega|\mathsf{T}\big(\Psi(x)\,\overline{\Psi}(y)\big)|\Omega\rangle \;=$$

$$+$$

$$+$$

$$+\;\mathcal{O}(q^4) \quad . \qquad (7.18a)$$

Evaluating these diagrams with the above Feynman rules, together with the 'sign rule' that each closed fermion loop comes with a factor of -1, one obtains

$$\begin{aligned}\langle\Omega|\mathsf{T}\big(\Psi(x)\,\overline{\Psi}(y)\big)|\Omega\rangle \;&=\; S_F(x-y) \\ &+ q^2\int_{\mathbb{R}^{2d}} S_F(x-z_1)\gamma^\mu S_F(z_1-z_2)\gamma^\nu S_F(z_2-y)\,(D_F)_{\mu\nu}(z_1-z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2 \\ &- q^2\int_{\mathbb{R}^{2d}} S_F(x-z_1)\gamma^\mu S_F(z_1-y)\;\mathrm{Tr}\big(S_F(0)\gamma^\nu\big)\;(D_F)_{\mu\nu}(z_1-z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2 \;+\;\mathcal{O}(q^4) \quad .\end{aligned} \qquad (7.18b)$$

To understand the origin of this sign rule, it is a good exercise to re-derive this result from a more rigorous algebraic perspective using the Gell-Mann and Low reduction formula (4.16) and Wick's theorem for boson and fermion fields. For the latter one has to pay attention to minus signs that arise from exchanging two fermionic field operators, see our discussion in Section 5.4.

Scattering amplitudes:

A new feature of the scattering amplitudes in QED is that the incoming and outgoing particles are polarized and the fermions also carry an electric charge. More precisely, a state $|k, s, \pm q\rangle$ associated with the Dirac field is labeled by the on-shell momentum $k \in \mathbb{R}^d$, spinor polarization s and charge $\pm q$ of the particle (see (5.84)) and a state $|k, \lambda\rangle$ associated with the photon field is labeled by the on-shell momentum $k \in \mathbb{R}^d$ and covector polarization λ of the photon (see (6.38)). The LSZ reduction formula (4.68) can be generalized to such types of particles, but due to time constraints we will not discuss the details, which can be found e.g. in the textbook by Greiner/Reinhardt (Exercise 9.2). The result of going through this tedious work is a list of Feynman rules for scattering amplitudes in QED. The momentum space QED Feynman rules for scattering amplitudes to lowest order perturbation theory (i.e. without loop diagram corrections) read as follows:

- Incoming fermion particle

$$|k, s, q\rangle \longrightarrow\!\bullet \quad = \quad u^s(\mathbf{k}) \tag{7.19}$$

- Outgoing fermion particle

$$\bullet\!\longrightarrow \langle k, s, q| \quad = \quad \overline{u^s(\mathbf{k})} \tag{7.20}$$

- Incoming fermion antiparticle

$$|k, s, -q\rangle \longleftarrow\!\bullet \quad = \quad \overline{v^s(\mathbf{k})} \tag{7.21}$$

- Outgoing fermion antiparticle

$$\bullet\!\longleftarrow \langle k, s, -q| \quad = \quad v^s(\mathbf{k}) \tag{7.22}$$

- Incoming photon

$$|k, \lambda\rangle \sim\!\bullet \quad = \quad \epsilon^\lambda(\mathbf{k}) \tag{7.23}$$

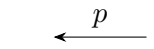
- Outgoing photon

$$\bullet\!\sim \langle k, \lambda| \quad = \quad \epsilon^\lambda(\mathbf{k}) \tag{7.24}$$

- Internal fermion line

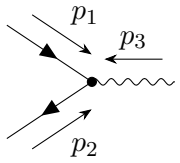
$$\overset{p}{\longleftarrow} \quad = \quad \frac{-\mathrm{i}\,(-\mathrm{i}\not{p} + m)}{p^2 + m^2 - \mathrm{i}\,\epsilon} \tag{7.25}$$

- Internal photon line



$$= \frac{-\mathrm{i}\,\eta_{\mu\nu}}{p^2 - \mathrm{i}\,\epsilon} \tag{7.26}$$

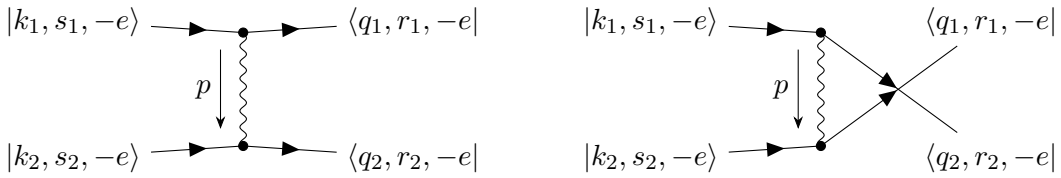
- Interaction vertex



$$= q\,\gamma^\mu\,(2\pi)^d\,\delta(p_1 + p_2 + p_3) \tag{7.27}$$

- Integration over internal momenta $\int_{\mathbb{R}^d} \frac{\mathrm{d}p}{(2\pi)^d}$
- A minus sign -1 for each fermion line crossing

Example 7.2 (Electron-electron scattering)**.** To develop some physical intuition, let's call the particle/antiparticle associated with Ψ the electron e^- and, respectively, the positron e^+. To lowest order in the coupling constant $q = -e$, the following two Feynman diagrams contribute to $e^-e^- \to e^-e^-$ scattering:



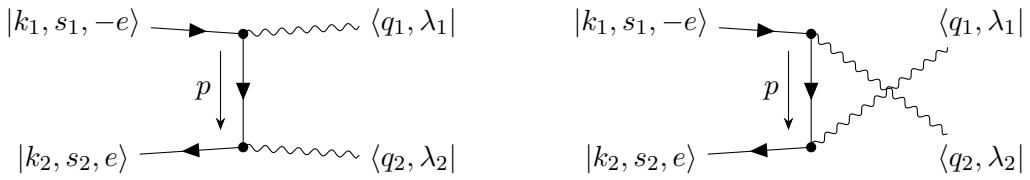
(7.28)

Using the Feynman rules, we determine the scattering amplitude

$$\begin{aligned}
&\langle (q_1, r_1, -e), (q_2, r_2, -e); \mathrm{out} | (k_1, s_1, -e), (k_2, s_2, -e); \mathrm{in} \rangle \\
&= e^2 \int_{\mathbb{R}^d} \Big(\overline{u^{r_1}(\mathbf{q}_1)} \gamma^\mu u^{s_1}(\mathbf{k}_1)\, \frac{-\mathrm{i}\,\eta_{\mu\nu}}{p^2 - \mathrm{i}\,\epsilon}\, \overline{u^{r_2}(\mathbf{q}_2)} \gamma^\nu u^{s_2}(\mathbf{k}_2)\, (2\pi)^{2d}\, \delta(k_1 - q_1 - p)\, \delta(k_2 - q_2 + p) \\
&\quad - \overline{u^{r_2}(\mathbf{q}_2)} \gamma^\mu u^{s_1}(\mathbf{k}_1)\, \frac{-\mathrm{i}\,\eta_{\mu\nu}}{p^2 - \mathrm{i}\,\epsilon}\, \overline{u^{r_1}(\mathbf{q}_1)} \gamma^\nu u^{s_2}(\mathbf{k}_2)\, (2\pi)^{2d}\, \delta(k_1 - q_2 - p)\, \delta(k_2 - q_1 + p) \Big) \frac{\mathrm{d}p}{(2\pi)^d} \\
&= -\mathrm{i}\, e^2\, (2\pi)^d\, \delta(k_1 + k_2 - q_1 - q_2) \left(\frac{\overline{u^{r_1}(\mathbf{q}_1)} \gamma^\mu u^{s_1}(\mathbf{k}_1)\; \overline{u^{r_2}(\mathbf{q}_2)} \gamma_\mu u^{s_2}(\mathbf{k}_2)}{(k_1 - q_1)^2} - \frac{\overline{u^{r_2}(\mathbf{q}_2)} \gamma^\mu u^{s_1}(\mathbf{k}_1)\; \overline{u^{r_1}(\mathbf{q}_1)} \gamma_\mu u^{s_2}(\mathbf{k}_2)}{(k_1 - q_2)^2} \right) .
\end{aligned} \tag{7.29}$$

This scattering amplitude contains a lot of physical details, such as how the strength of electron-electron interaction depends on the spinor polarizations (due to the numerators in the parenthesis) and on the momentum transfers $k_1 - q_1$ and $k_1 - q_2$ (due to the denominator). These fine details are for the moment not too relevant for us, but I would like to emphasize the following more conceptual observation: The electromagnetic interaction between two electrons is mediated by virtual photons, as one can see by looking at the Feynman diagrams above, and its strength is proportional to the product $(-e)\,(-e) = e^2$ of electric charges. With a more refined analysis, see e.g. the textbook by Nastase (Chapter 24.2), one can show that in the nonrelativistic limit this reproduces the well-known Coulomb potential between electric charges.

Example 7.3 (Electron-positron annihilation)**.** Another important process in QED is the annihilation of an electron/positron pair into a pair of photons. This is typically written as $e^- e^+ \to \gamma\gamma$, where γ refers to the physical name 'gamma ray/particle' for photons and not to a gamma matrix. To lowest order in the coupling constant $q = -e$, the following two Feynman diagrams contribute this scattering:



(7.30)

The scattering amplitude can be determined by using the Feynman rules, which I leave as an exercise for you.

Further reading

For more details about quantum electrodynamics, see e.g. Srednicki (Chapters 58 and 59), Maggiore (Chapter 7) Greiner/Reinhardt (Chapter 8.6) and Nastase (Chapter 23 onwards), from our reading list in Section 1.3.

Chapter 8

Renormalization

The perturbative approach to interacting QFT that we have developed in Chapter 4 is still incomplete and, even worse, seemingly inconsistent as a consequence of potential divergences in loop diagrams that contribute at higher orders in perturbation theory. See in particular Warning 4.5 for an illustration. This chapter provides a very brief introduction to *renormalization*, which is an important and successful technique that allows one to "cure" ultraviolet divergences by absorbing them into a suitable redefinition of the parameters of the QFT. To simplify our presentation, we consider only scalar QFTs, but it is important to stress that, with some additional computational efforts, other QFTs, e.g. quantum electrodynamics from Chapter 7, can be renormalized as well. Renormalization is *not* only a mathematical trick that leads to a well-defined perturbation theory, but it also has direct physical consequences, for instance in the form of "running" (i.e. energy-scale dependent) coupling constants. This can be formalized by so-called *renormalization group equations*, which is an advanced topic that we shall only touch very briefly in Remark 8.8. It is also important to stress that renormalization only deals with ultraviolet (i.e. short distance = large momenta) divergences, while the treatment of infrared (i.e. long distance = small momenta) divergences requires different techniques that will not be discussed in these lecture notes. To avoid the appearance of infrared divergences, we shall always work with a massive scalar field.

8.1 Superficial divergence and power counting

Before discussing the renormalization of ultraviolet divergences in detail, it will be useful to develop a heuristic but useful tool that allows us to identify the divergent Feynman diagrams. To simplify our presentation, we consider only the case of a scalar field theory on the d-dimensional Minkowski spacetime $(\mathbb{R}^d, \eta)$ with action

$$S[\Phi] = \int_{\mathbb{R}^d} -\left(\frac{1}{2}\,\partial^\mu\Phi\,\partial_\mu\Phi + \frac{m_0^2}{2}\,\Phi^2 + \frac{\lambda_0}{N!}\,\Phi^N\right)\mathrm{d}x \quad . \tag{8.1}$$

In this chapter it will be convenient to decorate the parameters m_0^2 and λ_0 of the classical action with a subscript $_0$ in order to distinguish them from the corresponding physical/renormalized parameters, see e.g. Observation 4.7 and Section 8.2 below. (In QFT jargon, the parameter m_0^2 is called the *bare mass* and λ_0 the *bare coupling constant*.) Associated with any Feynman diagram of

this theory (see Section 4.6 for the Feynman rules) are the following characteristic numbers:

$$E = \text{number of external lines} \quad , \tag{8.2a}$$
$$P = \text{number of internal lines/propagators} \quad , \tag{8.2b}$$
$$V = \text{number of vertices} \quad , \tag{8.2c}$$
$$L = \text{number of loops} \quad . \tag{8.2d}$$

From the Feynman rules, we see that each internal line comes with an undetermined momentum $l \in \mathbb{R}^d$, together with a momentum space integral $\int_{\mathbb{R}^d} \frac{\mathrm{d}l}{(2\pi)^d}$, and each vertex comes with a Dirac delta function $\delta(p_{\text{in}} - p_{\text{out}})$ that enforces relativistic momentum conservation at this vertex. The V many delta functions kill $V-1$ of the P many momentum space integrations, leaving one Dirac delta function that enforces overall relativistic momentum conservation. Hence, we obtain the identity

$$L = P - V + 1 \tag{8.3}$$

that expresses the number of loop integrals L in terms of P and V. Furthermore, since each external line ends with 1 end at a vertex and each internal line ends with 2 ends, we have

$$N\,V = 2\,P + E \quad , \tag{8.4}$$

where the N is due to the fact that Φ^N-theory has an N-valent interaction vertex. For large loop momenta $l_1, \dots, l_L \in \mathbb{R}^d$, the structure of the momentum space integral associated with our Feynman diagram looks schematically like

$$\int_{\mathbb{R}^{Ld}} \frac{1}{l_1^2 \cdots l_P^2}\, \mathrm{d}l_1 \cdots \mathrm{d}l_L \quad , \tag{8.5}$$

i.e. each loop momentum contributes a d-dimensional volume element $\mathrm{d}l$ and each internal line contributes a $1/l^2$. (Note that, for large loop momenta, the masses and external momenta in Feynman propagators can be neglected, e.g. $1/(l^2 + m_0^2) \approx 1/l^2$.) To obtain a quantity that allows for a rough estimate of the divergence/convergence property of such integrals, we compare the power of the momenta in the numerator and the denominator.

Definition 8.1. The *superficial degree of divergence* of a Feynman diagram is defined as

$$D := d\,L - 2\,P \quad . \tag{8.6}$$

Continuing with our heuristic analysis, we introduce a momentum space cut-off, i.e. we integrate only over momenta whose norm is less than some constant $\Lambda > 0$, and thereby find the following behavior

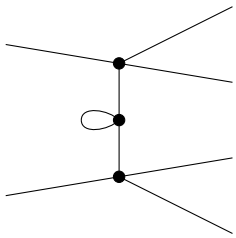
$$\int_{\mathbb{R}^{Ld}}^{\Lambda} \frac{1}{l_1^2 \cdots l_P^2}\, \mathrm{d}l_1 \cdots \mathrm{d}l_L \sim \begin{cases} \Lambda^D & , \text{ for } D > 0 \quad , \\ \log(\Lambda) & , \text{ for } D = 0 \quad , \\ \Lambda^{-|D|} & , \text{ for } D < 0 \quad . \end{cases} \tag{8.7}$$

This heuristic power counting analysis can be made more precise by using the concept of Wick rotation (see around Eqn. (8.59) below) to turn the integral (8.7) into an Euclidean momentum

space integral. This extra level of precision is however not necessary for understanding the main message of the present section. According to this heuristic reasoning, one finds that Feynman diagrams with $D \geq 0$ are divergent when sending the cut-off $\Lambda \to \infty$, while those with $D < 0$ are finite. Unfortunately, the superficial degree of divergence D does not always reflect the actual divergence or finiteness property of a Feynman diagram, scattering amplitude or time-ordered n-point function. Indeed, the following exceptions might happen:

1. In the superficially divergent case $D \geq 0$, infinite terms might cancel each other and lead to a finite end result. This typically happens in gauge theories or in QFTs with supersymmetry.

2. In the superficially finite case $D < 0$, a diagram might be divergent if the divergence is due to a simpler divergent subdiagram. A simple example for this phenomenon is given by the Feynman diagram



(8.8)

in Φ^4-theory in $d = 4$ dimensions. This diagram has $L = 1$ loops and $P = 3$ internal lines, hence $D = 4 \times 1 - 2 \times 3 = -2$. However, as we shall see in Section 8.3, the tadpole loop on the internal propagator is divergent, hence the overall diagram is divergent too.

The first exception is pretty harmless, because it simply means that the superficial degree of divergence might overestimate the number of divergent diagrams. The second exception is potentially quite serious, because it means that, by looking only at the superficial degree of divergence, we might miss some of the divergent diagrams. While a general treatment of subdivergences of Feynman diagrams is possible, it is unfortunately rather technical and hence beyond the scope of this module. In our simple example (8.8), the subdivergence of this diagram will simply be removed by renormalizing the interacting Feynman propagator.

Leveraging the identities (8.3) and (8.4), we can express the superficial degree of divergence (8.6) in the following more useful form

$$D \;=\; d + \left(N\,\tfrac{d-2}{2} - d\right) V - \tfrac{d-2}{2}\,E \tag{8.9}$$

in terms of the number of vertices V and the number of external lines E. From this expression one sees that the prefactor $N\,\frac{d-2}{2} - d$ of V is crucial: If this prefactor is negative, then Feynman diagrams with sufficiently many vertices become superficially finite (i.e. $D < 0$), hence the QFT has only a finite number of superficially divergent diagrams. On the other hand, if this prefactor is positive, then *all* Feynman diagrams with a sufficiently high number of vertices V will become superficially divergent (i.e. $D \geq 0$). If the prefactor of V is 0 (which can only happen for $d \geq 3$), then D does not depend on the number of vertices but only on the number E of external lines. Since the prefactor of E is negative in this case, we have that $D < 0$ for sufficiently large E. This means that only finitely many scattering amplitudes and/or time-ordered n-point functions, namely those

with a sufficiently small number E of external lines, are superficially divergent. It is important to distinguish between these three different possibilities, which is done by introducing the following terminology.

Definition 8.2. A QFT is called

- (power counting) *super-renormalizable* if only a finite number of connected Feynman diagrams superficially diverge,
- (power counting) *renormalizable* if only a finite number of time-ordered n-point functions and/or scattering amplitudes superficially diverge (note that these can consist of infinitely many superficially divergent connected Feynman diagrams),
- (power counting) *non-renormalizable* if infinitely many time-ordered n-point functions and/or scattering amplitudes are superficially divergent at a sufficiently high order in perturbation theory.

Note that

$$\text{super-renormalizable} \quad \Longrightarrow \quad \text{renormalizable} \quad , \tag{8.10}$$

hence super-renormalizable QFTs are a (very special) subclass of the renormalizable ones.

Remark 8.3. There is a huge qualitative difference between renormalizable and non-renormalizable QFTs. For the former type there are only finitely many divergent time-ordered n-point functions and/or scattering amplitudes, while for the latter there are infinitely many. Using the renormalization techniques to be developed in this chapter, we will be able to "absorb" these divergences into a redefinition of the parameters of the QFT. For renormalizable QFTs, this will result in a finite number of free parameters, which have to be fixed from experimental input. On the other hand, for non-renormalizable QFTs, absorbing the infinitely many divergences will result in an infinite number of free parameters, which makes such theories non-predictive as all these parameters must be fixed by experiment. As a consequence, physicists are typically looking for renormalizable QFTs to describe nature, but also non-renormalizable QFTs appear in certain areas of low-energy physics as "effective QFTs".

Warning 8.4. As already mentioned above, the superficial degree of divergence does not necessarily reflect the actual divergence or finiteness properties of a QFT. This means that Definition 8.2 of (power counting) super-renormalizable, renormalizable and non-renormalizable QFTs should be used with some care. In my opinion, this definition is best used as a guiding principle to propose and design suitable QFT models which have a chance to be renormalizable in a mathematically strict sense. For many physically relevant QFTs, such as QED and Yang-Mills theory in $d = 4$ dimensions, there exist rigorous proofs confirming their renormalizability.

Example 8.5 (Spacetime dimension $d = 4$)**.** Let us consider the physically relevant case of $d = 4$ spacetime dimensions. Then the superficial degree of divergence in (8.9) specializes to

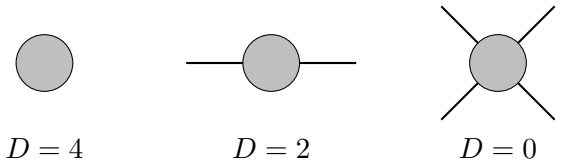
$$D \;=\; 4 + (N-4)\, V - E \quad . \tag{8.11}$$

From this we find that Φ^3-theory (i.e. $N = 3$) in $d = 4$ dimensions is super-renormalizable and that Φ^4-theory (i.e. $N = 4$) in $d = 4$ dimensions is renormalizable. In contrast to this, Φ^N-theory in $d = 4$ dimensions is non-renormalizable for all $N \geq 5$.

Let us discuss the case of Φ^4-theory in $d = 4$ dimensions in more detail. Since the action functional

$$S[\Phi] = \int_{\mathbb{R}^4} -\Big(\frac{1}{2}\,\partial^\mu\Phi\,\partial_\mu\Phi + \frac{m_0^2}{2}\,\Phi^2 + \frac{\lambda_0}{4!}\,\Phi^4\Big)\,\mathrm{d}x \tag{8.12}$$

is invariant under the sign flip transformation $\Phi \mapsto -\Phi$, all odd time-ordered n-point functions must vanish. Using the superficial degree of divergence $D = 4 - E$ for this theory, we further see that all time-ordered n-point functions with $n = E \geq 5$ are superficially finite. So the only superficially divergent time-ordered n-point functions are the 0-, 2- and 4-point functions



(8.13)

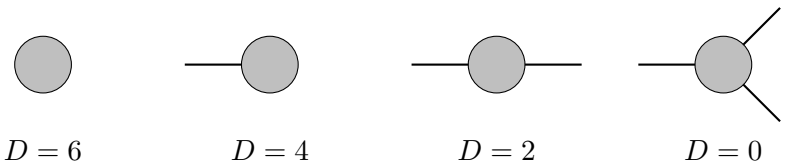
The divergences in the time-ordered 0-point function are given by vacuum bubbles, hence they describe an unobservable vacuum energy shift which can be ignored. The divergences in the 2-point function will lead to a renormalization of the mass parameter (see Observation 4.7) and, at higher loop orders, also a wave function renormalization. Finally, the divergences in the 4-point function will lead to a renormalization of the coupling constant. One of the goals of this chapter is to make these claims precise.

Example 8.6 (Spacetime dimension $d = 6$)**.** The superficial degree of divergence (8.9) specializes in $d = 6$ dimensions to

$$D = 6 + (2\,N - 6)\,V - 2\,E \quad . \tag{8.14}$$

From this we find that Φ^3-theory (i.e. $N = 3$) in $d = 6$ dimensions is renormalizable and that Φ^N-theory in $d = 6$ dimensions is non-renormalizable for all $N \geq 4$. Comparing with the previous example in $d = 4$, we note that it becomes harder to find renormalizable QFTs when increasing the dimension of spacetime.

Let us discuss the case of Φ^3-theory in $d = 6$ dimensions in more detail. Using the superficial degree of divergence $D = 6 - 2\,E$ for this theory, we see that all time-ordered n-point functions with $n = E \geq 4$ are superficially finite. So the only superficially divergent time-ordered n-point functions are the 0-, 1-, 2- and 3-point functions



(8.15)

Remark 8.7. The relevant prefactor $N\,\frac{d-2}{2} - d$ of V in (8.9) is related to the physical dimension of the coupling constant λ_0 in the action (8.1). Since we work in natural units $c = 1$ and $\hbar = 1$, we can measure all quantities (length, time, momentum, energy, ...) in the dimension of mass. For example, the mass dimension of the spacetime coordinates is $[x^\mu] = -1$, which implies that

the d-dimensional volume element has mass dimension $[\mathrm{d}x] = -d$, and the one of the derivative is $[\partial_\mu] = 1$. Since the action must be dimensionless, we find that the scalar field has mass dimension

$$[\Phi] = \frac{d-2}{2} \quad . \tag{8.16}$$

For the parameters in the action, we then find $[m_0^2] = 2$, as expected, and

$$[\lambda_0] = d - N\,\frac{d-2}{2} \quad , \tag{8.17}$$

which is the additive inverse of the prefactor $N\,\frac{d-2}{2} - d$ of V in (8.9). Summing up, this means that super-renormalizable theories have coupling constants with positive mass dimension $[\lambda_0] > 0$, renormalizable theories have dimensionless coupling constants $[\lambda_0] = 0$ and non-renormalizable theories have coupling constants with negative mass dimension $[\lambda_0] < 0$. This observation is useful in practice when designing examples of action functionals for physical theories, because via this simple dimensional analysis one can get a good hint whether the resulting QFT will be renormalizable or not.

8.2 Renormalized perturbation theory

In Observation 4.7, we have already seen indications that the *bare* parameters m_0^2 and λ_0 from the classical action (8.1) receive quantum corrections from loop diagrams and hence they do *not* describe the physical parameters of the QFT. Furthermore, as indicated in Observation 4.8, the field Φ itself will in general receive quantum corrections to its normalization. *Renormalized perturbation theory* is a clever and practically very useful reformulation of the perturbation theory from Chapter 4 in which the action is expressed in terms of the physical/renormalized parameters, instead of the bare ones, and the renormalized field.

Let us explain how this works for Φ^4-theory in $d = 4$ spacetime dimensions, which is our prime example for a renormalizable QFT, see Example 8.5. Other examples of renormalizable QFTs, such as Φ^3-theory in $d = 6$ dimensions from Example 8.6, can be treated similarly. Let us first take care of the wave function renormalization by defining the *renormalized field* as

$$\Phi_r \;:=\; Z^{-\frac{1}{2}}\,\Phi \quad . \tag{8.18}$$

This rescaling is chosen to cancel the Z-factor in the interacting propagator, see Observation 4.8. (As a side-remark, note that this rescaling implies that, when working with Φ_r, there are no Z-factors in the LSZ formula (4.68).) Inserting $\Phi = Z^{\frac{1}{2}}\,\Phi_r$ into the classical action (8.12) of d=4-dimensional Φ^4-theory, we obtain

$$S[\Phi_r] \;=\; \int_{\mathbb{R}^4} -\Big(\frac{Z}{2}\,\partial^\mu\Phi_r\,\partial_\mu\Phi_r + \frac{m_0^2\,Z}{2}\,\Phi_r^2 + \frac{\lambda_0\,Z^2}{4!}\,\Phi_r^4\Big)\,\mathrm{d}x \quad . \tag{8.19}$$

The bare mass and coupling constant still appear in this action, but they can be eliminated as follows. Let us introduce

$$\delta_Z \;:=\; Z - 1 \quad , \quad \delta_m \;:=\; m_0^2\,Z - m^2 \quad , \quad \delta_\lambda \;:=\; \lambda_0\,Z^2 - \lambda \quad , \tag{8.20}$$

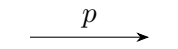
where m^2 and λ are the physically measured mass and coupling constant. Inserting this into the action, we obtain

$$S[\Phi_r] = \int_{\mathbb{R}^4} -\Big(\frac{1}{2}\,\partial^\mu\Phi_r\,\partial_\mu\Phi_r + \frac{m^2}{2}\,\Phi_r^2 + \frac{\lambda}{4!}\,\Phi_r^4 + \frac{\delta_Z}{2}\,\partial^\mu\Phi_r\,\partial_\mu\Phi_r + \frac{\delta_m}{2}\,\Phi_r^2 + \frac{\delta_\lambda}{4!}\,\Phi_r^4\Big)\,\mathrm{d}x \quad . \tag{8.21}$$

The terms in the first line are the familiar ones from Φ^4-theory, but importantly they are expressed in terms of the physical/renormalized field Φ_r, mass m^2 and coupling λ. The terms in the second line are called the *counterterms* and, as we shall see later in this chapter, their role is to absorb the divergences in the renormalization process. So summing up, the advantage of working with the action in the form of (8.21) is that there is a clear split between the physical quantities, which must be finite, and the unphysical counterterms, which can be infinite due to divergences from loop diagrams.

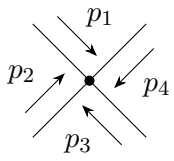
The counterterms in the action (8.21) lead to additional Feynman rules on top of the ones from Section 4.6. For example, for scattering amplitudes in momentum space, the relevant Feynman rules are as follows:

- Propagator



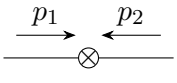
$$= \frac{-\mathrm{i}}{p^2 + m^2 - \mathrm{i}\,\epsilon} \tag{8.22}$$

- Interaction vertex



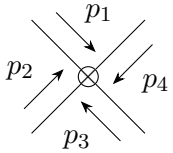
$$= -\mathrm{i}\,\lambda\,(2\pi)^4\,\delta(p_1 + p_2 + p_3 + p_4) \tag{8.23}$$

- Propagator counterterm



$$= -\mathrm{i}\,(p_1^2\,\delta_Z + \delta_m)\,(2\pi)^4\,\delta(p_1 + p_2) \tag{8.24}$$

- Vertex counterterm



$$= -\mathrm{i}\,\delta_\lambda\,(2\pi)^4\,\delta(p_1 + p_2 + p_3 + p_4) \tag{8.25}$$

All of the above is meaningless until we state very clearly how the physical/renormalized parameters are linked to suitable quantum field theoretical quantities, such as time-ordered n-point functions or scattering amplitudes. It is important to emphasize that perturbative QFT is *not* capable to compute these parameters from first principles, but they rather have to be fixed via experimental input. Creating a link between the experimentally determined physical parameters and quantum field theoretical quantities is however still necessary in order to fix the so far undetermined counterterms δ_Z, δ_m and δ_λ in the action (8.21). In QFT jargon, such links are called *renormalization conditions*. We shall now discuss a particularly simple, but useful, set of renormalization conditions which is called the *on-shell renormalization scheme*. (There exist other renormalization schemes, e.g. the MS-scheme and MS-bar-scheme, but these will not be discussed in these lecture notes.) The idea behind the on-shell renormalization scheme is quite simple: To link our QFT to the experimentally determined physical mass m^2, we consider the Fourier transform of the interacting Feynman propagator

$$\langle\Omega|\mathsf{T}\big(\Phi_r(x)\,\Phi_r(y)\big)|\Omega\rangle = \int_{\mathbb{R}^4} \widetilde{\Delta}_F^{\text{int}}(k)\, e^{\mathrm{i}\,k\,(x-y)}\, \frac{\mathrm{d}k}{(2\pi)^4} \tag{8.26}$$

of the renormalized field Φ_r. (In the next section we will compute $\widetilde{\Delta}_F^{\text{int}}(k)$ at the 1-loop level using our Feynman rules with counterterms from above.) The physical mass m^2 is then identified as the pole of $\widetilde{\Delta}_F^{\text{int}}(k)$, i.e.

$$\widetilde{\Delta}_F^{\text{int}}(k) \;=\; \frac{-\mathrm{i}}{k^2+m^2-\mathrm{i}\,\epsilon} \;+\; (\text{terms regular at } k^2=-m^2) \quad . \tag{8.27}$$

Note that (8.27) demands two independent conditions, namely that the pole of $\widetilde{\Delta}_F^{\text{int}}(k)$ is located at $k^2=-m^2$ and that its residue is $-\mathrm{i}$. Inverting $\widetilde{\Delta}_F^{\text{int}}(k)$, we can disentangle these two conditions into the following two renormalization conditions:

(RC1) $$\widetilde{\Delta}_F^{\text{int}}(k)^{-1}\Big|_{k^2=-m^2} \;=\; 0 \quad , \tag{8.28}$$

(RC2) $$\frac{\mathrm{d}}{\mathrm{d}k^2}\widetilde{\Delta}_F^{\text{int}}(k)^{-1}\Big|_{k^2=-m^2} \;=\; \mathrm{i} \quad . \tag{8.29}$$

In the next section we will see, by an explicit computation at the 1-loop level, that these two renormalization conditions fix the two counterterms δ_Z and δ_m. In order to link our QFT to the experimentally determined physical coupling λ, let us consider the $2\to 2$ scattering amplitude

$$\langle q_1,q_2;\text{out}|k_1,k_2;\text{in}\rangle \;=:\; \mathcal{A}(k_1,k_2,q_1,q_2)\,(2\pi)^4\,\delta(k_1+k_2-q_1-q_2) \quad , \tag{8.30}$$

where, for later convenience, we have factorized off the Dirac delta function that enforces overall relativistic momentum conservation. Motivated by the fact that, at lowest order perturbation theory, the amplitude $\mathcal{A}(k_1,k_2,q_1,q_2)$ is proportional to the coupling constant via (4.78), it makes sense to formulate a renormalization condition that links the physical coupling λ to $\mathcal{A}(k_1,k_2,q_1,q_2)$. Since the amplitude (including quantum corrections) will in general depend on the momenta, we have to specify an energy scale at which we link $\mathcal{A}(k_1,k_2,q_1,q_2)$ to the experimental value of λ. The relevant energy scales appearing in $2\to 2$ scattering are given by the three *Mandelstam variables*

$$s \;:=\; (k_1+k_2)^2 \quad , \quad t \;:=\; (k_1-q_1)^2 \quad , \quad u \;:=\; (k_1-q_2)^2 \quad , \tag{8.31}$$

which (in disguise) already appeared in (4.80), even though we didn't call them s, t, u yet. Assuming that our experiment took place at very low energies, e.g. in a normal lab and not in a particle collider, all particles are slow so that their rest energy dominates their spatial momenta. This means that we have approximately $k_1 \approx k_2 \approx q_1 \approx q_2 \approx (m, 0, 0, 0)$, hence $s = -4m^2$, $t = 0$ and $u = 0$. In this scenario it makes sense to introduce the following renormalization condition

$$\text{(RC3)} \qquad \mathcal{A}(k_1, k_2, q_1, q_2)\Big|_{s=-4m^2, t=u=0} = -\mathrm{i}\,\lambda \quad , \tag{8.32}$$

which links the physical coupling constant to the low-energy limit of the scattering amplitude. In the next section we will see, by an explicit computation at the 1-loop level, that this renormalization condition fixes the counterterm δ_λ.

The key feature of renormalizable QFTs is that, once the finitely many counterterms are fixed by finitely many renormalization conditions in terms of finitely many experimentally determined parameters, *all* time-ordered n-point functions and scattering amplitudes become finite. In particular, after renormalization has been carried out, the QFT does not have any divergences anymore and it becomes useful for physical predictions. The heuristic explanation for this lies in our analysis of the superficial degree of divergence in Section 8.1, but with some more efforts, which go beyond the scope of this module, one can also prove this rigorously for many examples of QFTs, such as Φ^4-theory, QED and Yang-Mills theory in $d = 4$ spacetime dimensions.

Remark 8.8. Our choice (8.32) to use the low-energy scattering amplitude to formulate a renormalization condition for the coupling constant λ is clearly arbitrary, and so are the other two renormalization conditions in (8.28) and (8.29). We could have equally well formulated renormalization conditions at some other energy scale μ, the so-called *renormalization scale*, in order to relate the physical parameters, which now have to be determined experimentally at the energy scale μ, and the counterterms. Since these are just different ways to define the same QFT, it should not matter at which energy scale μ one imposes renormalization conditions. This physical expectation can be realized mathematically as a family of differential relations, called the *Callan-Symanzik equations*, for the interacting time-ordered n-point functions

$$G_n(x_1, \dots, x_n) := \langle \Omega | \mathsf{T}(\Phi_r(x_1) \cdots \Phi_r(x_n)) | \Omega \rangle \quad . \tag{8.33}$$

In the simplest case of a *massless* Φ^4-theory in $d = 4$ dimensions, the Callan-Symanzik equations read as

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} + n\,\gamma\right) G_n(x_1, \dots, x_n) = 0 \qquad \text{(for all non-negative integers } n) \quad , \tag{8.34a}$$

where

$$\beta := \mu \frac{\mathrm{d}\lambda}{\mathrm{d}\mu} \quad , \qquad \gamma := \mu \frac{\mathrm{d} \log\left(Z^{\frac{1}{2}}\right)}{\mathrm{d}\mu} \tag{8.34b}$$

are the so-called *β-function* and *anomalous dimension*, which are independent of n. See e.g. Peskin/Schroeder (Chapter 12.2) for a derivation. The interacting time-ordered n-point functions $G_n(x_1, \dots, x_n)$ can be computed via similar perturbative QFT techniques as in Section 8.3 below.

Performing these calculations at the 1-loop level and inserting the results into the Callan-Symanzik equations, one finds

$$\beta \,=\, \mu\,\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} \,=\, \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3) \quad , \qquad \gamma \,=\, \mu\,\frac{\mathrm{d}\log\left(Z^{\frac{1}{2}}\right)}{\mathrm{d}\mu} \,=\, 0 + \mathcal{O}(\lambda^2) \quad . \tag{8.35}$$

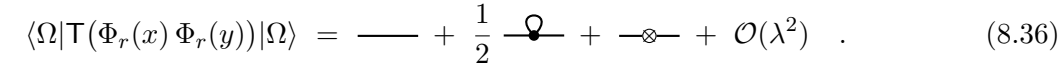
This implies that, at the 1-loop level, the wave function renormalization is independent of the renormalization scale μ, however the physical coupling constant *does* depend on the energy μ, hence one should better write $\lambda(\mu)$. (In QFT jargon, one calls this a *"running" coupling constant.*) Since the β-function is positive, it follows that $\lambda(\mu)$ grows when we increase the energy scale μ, i.e. the interactions of Φ^4-theory in $d = 4$ dimensions are weaker at low energies and stronger at high energies. This particular behavior of the "running" coupling constant is however not universal, i.e. it depends on details of the QFT. Other QFTs, such as quantum chromodynamics (QCD), have a negative β-function, hence their coupling constants decrease when increasing the energy scale μ, and sometimes they even run to 0 in the ultra-high energy limit $\mu \to \infty$. The latter phenomenon is called *asymptotic freedom* and its discovery by Gross, Wilczek and Politzer was rewarded with the 2004 Nobel Prize in Physics.

8.3 1-loop renormalization

In this section we shall make precise in which sense the renormalization conditions allow us to determine the counterterms. For illustrative purposes, we consider again the example of Φ^4-theory in $d = 4$ dimensions, see Section 8.2 for the relevant Feynman rules including counterterms, and focus only on 1-loop corrections. Renormalization at higher orders in perturbation theory is possible but computationally much more involved. If you are interested in these aspects, you can have a look at the book by Peskin/Schroeder (Chapters 10.4 and 10.5), but this material is not relevant for our module.

Fixing the counterterms: Let us focus first on the two renormalization conditions (8.28) and (8.29) for the interacting Feynman propagator. Following similar arguments as in (4.41), we can use perturbation theory (in the form of the Gell-Mann and Low formula and Wick's theorem) to determine the Feynman diagrams for the interacting propagator and find

$\langle\Omega|\mathsf{T}\big(\Phi_r(x)\,\Phi_r(y)\big)|\Omega\rangle$ =



$+ \mathcal{O}(\lambda^2)$. (8.36)

Note that the counterterm in the last diagram is a new feature of renormalized perturbation theory that wasn't previously there in (4.41) where we worked with the bare parameters. The Fourier transform of the interacting Feynman propagator can be determined with a similar calculation as in (4.43) and one finds

$$\widetilde{\Delta}_F^{\mathrm{int}}(k) \,=\, \frac{-\mathrm{i}}{k^2 + m^2 + \frac{\lambda}{2}\,\Delta_F(0) + \delta_Z\,k^2 + \delta_m - \mathrm{i}\,\epsilon} \,+\, \mathcal{O}(\lambda^2) \quad . \tag{8.37}$$

Inverting this expression, we can send $\epsilon \to 0$ and obtain

$$\widetilde{\Delta}_F^{\mathrm{int}}(k)^{-1} \,=\, \mathrm{i}\left(k^2 + m^2 + \frac{\lambda}{2}\,\Delta_F(0) + \delta_Z\,k^2 + \delta_m\right) \,+\, \mathcal{O}(\lambda^2) \quad . \tag{8.38}$$

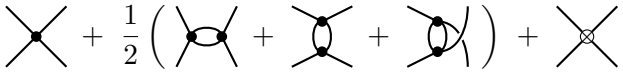
The two renormalization conditions (8.28) and (8.29) then fix the two counterterms according to

$$\delta_Z \,=\, 0 + \mathcal{O}(\lambda^2) \quad , \quad \delta_m \,=\, -\frac{\lambda}{2}\,\Delta_F(0) \,+\, \mathcal{O}(\lambda^2) \quad , \tag{8.39}$$

where the loop integral

$$\Delta_F(0) \,=\, \lim_{\epsilon\to 0} \int_{\mathbb{R}^4} \frac{-\mathrm{i}}{l^2 + m^2 - \mathrm{i}\,\epsilon}\,\frac{\mathrm{d}l}{(2\pi)^4} \tag{8.40}$$

is given explicitly by the formula (3.57) for the Feynman propagator. We postpone the computation of this loop integral to a later point in this section and focus next on the third renormalization condition (8.32). The $2 \to 2$ scattering amplitude can be computed perturbatively using the Feynman rules with counterterms from Section 8.2 (see also Section 4.6 for further details) and one finds

$\langle q_1, q_2; \mathrm{out} | k_1, k_2; \mathrm{in}\rangle \,=$  $+ \mathcal{O}(\lambda^3)$. (8.41)

Note that the three loop diagrams are all similar and related to each other by exchanging the external momenta k_1, k_2, q_1, q_2. Looking for instance at the first loop diagram, we can compute via the Feynman rules

 $= (-\mathrm{i}\,\lambda)^2 \left(\displaystyle\int_{\mathbb{R}^4} \frac{-\mathrm{i}}{l^2 + m^2 - \mathrm{i}\,\epsilon}\,\frac{-\mathrm{i}}{(k_1 + k_2 - l)^2 + m^2 - \mathrm{i}\,\epsilon}\,\frac{\mathrm{d}l}{(2\pi)^4} \right) (2\pi)^4\,\delta(k_1 + k_2 - q_1 - q_2)$. (8.42)

Denoting the $\epsilon \to 0$ limit of this loop integral by

$$\mathrm{i}\,V(p^2) \,:=\, \lim_{\epsilon\to 0} \int_{\mathbb{R}^4} \frac{-\mathrm{i}}{l^2 + m^2 - \mathrm{i}\,\epsilon}\,\frac{-\mathrm{i}}{(p - l)^2 + m^2 - \mathrm{i}\,\epsilon}\,\frac{\mathrm{d}l}{(2\pi)^4} \quad , \tag{8.43}$$

we obtain for the scattering amplitude

$$\langle q_1, q_2; \mathrm{out} | k_1, k_2; \mathrm{in}\rangle \,=\, \mathcal{A}(k_1, k_2, q_1, q_2)\; (2\pi)^4\,\delta(k_1 + k_2 - q_1 - q_2) \tag{8.44a}$$

the following expression

$$\begin{aligned} \mathcal{A}(k_1, k_2, q_1, q_2) \,&=\, -\mathrm{i}\,\lambda + \frac{(-\mathrm{i}\,\lambda)^2}{2}\Big(\mathrm{i}\,V(s) + \mathrm{i}\,V(t) + \mathrm{i}\,V(u)\Big) - \mathrm{i}\,\delta_\lambda \,+\, \mathcal{O}(\lambda^3) \\ &=\, -\mathrm{i}\left(\lambda + \frac{\lambda^2}{2}\Big(V(s) + V(t) + V(u)\Big) + \delta_\lambda\right) + \mathcal{O}(\lambda^3) \quad , \end{aligned} \tag{8.44b}$$

where we recall that s, t, u denote the three Mandelstam variables (8.31). Our third renormalization condition (8.32) then fixes the counterterm according to

$$\delta_\lambda \,=\, -\frac{\lambda^2}{2}\Big(V(-4m^2) + 2\,V(0)\Big) \,+\, \mathcal{O}(\lambda^3) \quad . \tag{8.45}$$

Summing up, with equations (8.39) and (8.45) we achieved our goal to fix all three counterterms (δ_Z, δ_m and δ_λ) in terms of loop integrals and the given physical parameters.

Towards computing the loop integrals: It remains to determine the loop integrals in (8.40) and (8.43) that enter our counterterms above. Since these integrals are divergent (at least superficially, as one easily checks by power counting), one should not expect to find a numerical value, but rather one should analyze their divergent behavior. In order to do so, one has to introduce a suitable *regulator* that makes these integrals finite and characterize which kinds of divergences arise in the limit of removing the regulator. The simplest regulator is a *momentum space cut-off*, i.e. one restricts the integrations to loop momenta whose norm is less than some constant $\Lambda > 0$, and then investigates the divergences arising in the limit $\Lambda \to \infty$. (Recall that we have done something like this in our heuristic study of the superficial degree of divergence, see (8.7).) Since momentum space cut-offs are cumbersome to work with, and also not suitable for gauge theories because they break gauge invariance, QFT practitioners have invented smarter regularization techniques, such as *dimensional regularization* which we shall use below. The basic idea behind dimensional regularization is to consider an analytic extension of the loop integrals to complex spacetime dimensions $d \in \mathbb{C}$, where they become finite, and then study the divergences that arise when sending the dimension $d \to 4$ or, more generally, to the desired integer dimension one is interested in. This may sound a bit crazy, but as we shall see below this method is very convenient for QFT computations. Please note that dimensional regularization is just a mathematical trick to regularize a QFT! One should not philosophize about what one means by physics in dimension $d = \sqrt{2} + \mathrm{i}\,\pi$.

Dimensional regularization is a technique that applies to d-dimensional integrals of the form

$$I_{d,n}(\Delta) := \int_{\mathbb{R}^d} \frac{1}{(l^2+\Delta)^n} \frac{\mathrm{d}l}{(2\pi)^d} \quad , \tag{8.46}$$

where Δ is some complex parameter. While our first loop integral (8.40) is clearly of this type (set $n = 1$ and $\Delta = m^2 - \mathrm{i}\,\epsilon$), it is not immediately obvious that our second loop integral (8.43) is too. There is a nice trick, going back to Feynman, that allows us to rewrite (8.43) as an integral of the form (8.46). This trick is based on the following observation.

Lemma 8.9 (Feynman parametrization)**.** *Let $A, B \in \mathbb{C}$ be two complex numbers such that 0 is not contained in the line segment connecting A and B, i.e. $B + \alpha\,(A - B) \neq 0$ for all $\alpha \in [0, 1]$. Then the identity*

$$\frac{1}{A\,B} = \int_0^1 \frac{1}{\big(\alpha\,A + (1-\alpha)\,B\big)^2}\,\mathrm{d}\alpha \tag{8.47}$$

holds true.

Proof. The following proof is from the Wikipedia article. For your convenience, it will be repeated here. We can write

$$\frac{1}{A\,B} = \frac{1}{A-B}\left(\frac{1}{B} - \frac{1}{A}\right) = \frac{1}{A-B}\int_B^A \frac{\mathrm{d}z}{z^2} \quad . \tag{8.48}$$

Using now the substitution $\alpha = (z-B)/(A-B)$, we have $\mathrm{d}\alpha = \mathrm{d}z/(A-B)$ and $z = \alpha\,A + (1-\alpha)\,B$, from which the claim follows. □

Remark 8.10. This result can be generalized to more factors, which yields the identity

$$\frac{1}{A_1\,A_2\cdots A_n} \;=\; (n-1)!\int_{[0,1]^n}\frac{\delta(\alpha_1+\alpha_2+\cdots+\alpha_n-1)}{\big(\alpha_1\,A_1+\alpha_2\,A_2+\cdots+\alpha_n\,A_n\big)^n}\;\mathrm{d}\alpha_1\cdots\mathrm{d}\alpha_n \quad . \tag{8.49}$$

This general form of Feynman parametrization turns out to be useful to compute loop integrals that involve n many propagators.

Using Lemma 8.9, we can rewrite the d-dimensional generalization of our second loop integral (8.43) as follows:

$$\begin{aligned}
\mathrm{i}\,V(p^2) \;&=\; -\lim_{\epsilon\to 0}\int_{\mathbb{R}^d}\frac{1}{l^2+m^2-\mathrm{i}\,\epsilon}\,\frac{1}{(p-l)^2+m^2-\mathrm{i}\,\epsilon}\,\frac{\mathrm{d}l}{(2\pi)^d}\\
&=\; -\lim_{\epsilon\to 0}\int_{\mathbb{R}^d}\int_0^1\frac{1}{\big(l^2-2\,(1-\alpha)\,p\,l+(1-\alpha)\,p^2+m^2-\mathrm{i}\,\epsilon\big)^2}\;\mathrm{d}\alpha\,\frac{\mathrm{d}l}{(2\pi)^d}\\
&=\; -\lim_{\epsilon\to 0}\int_{\mathbb{R}^d}\int_0^1\frac{1}{\big(k^2+\alpha\,(1-\alpha)\,p^2+m^2-\mathrm{i}\,\epsilon\big)^2}\;\mathrm{d}\alpha\,\frac{\mathrm{d}k}{(2\pi)^d}\\
&=\; -\lim_{\epsilon\to 0}\int_0^1\left(\int_{\mathbb{R}^d}\frac{1}{\big(l^2+\alpha\,(1-\alpha)\,p^2+m^2-\mathrm{i}\,\epsilon\big)^2}\,\frac{\mathrm{d}l}{(2\pi)^d}\right)\mathrm{d}\alpha \quad .
\end{aligned} \tag{8.50}$$

In the third step we substituted $k = l-(1-\alpha)\,p$ and in the last step we relabeled k back to l and also exchanged the order of integrations. (Note that there is no theorem telling us that we are allowed to exchange the order of integrations, but this step is part of the regularization process.) The integral in the parenthesis is of the form (8.46) with $n=2$ and $\Delta = \alpha\,(1-\alpha)\,p^2+m^2-\mathrm{i}\,\epsilon$.

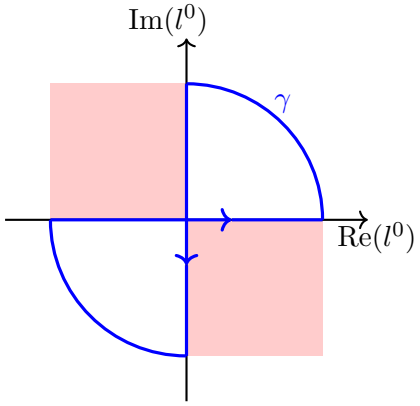
The master integral (8.46): Since our desired loop integrals (8.40) and (8.50) are special cases of (8.46), it remains to be shown how this general integral can be analyzed. Looking at the integrand, we note that it has poles at

$$l^2+\Delta = -(l^0)^2+\mathbf{l}^2+\Delta \;=\; 0 \quad\Longleftrightarrow\quad l^0 = \pm\sqrt{\mathbf{l}^2+\Delta} \quad . \tag{8.51}$$

To understand the location of these poles in the complex plane, we note that in all our examples above the imaginary part of Δ is negative, i.e. $\mathrm{Im}(\Delta)<0$, as a consequence of the $-\mathrm{i}\,\epsilon$ terms in the Feynman propagator. Hence, also the imaginary part of $\mathbf{l}^2+\Delta$ is negative because $\mathbf{l}^2$ is real. Writing this complex number in polar form, we get $\mathbf{l}^2+\Delta = |\mathbf{l}^2+\Delta|\;e^{\mathrm{i}\,\phi}$ with a phase $\phi\in(\pi,2\pi)$. The two square roots of this complex number are then given by

$$\pm\sqrt{\mathbf{l}^2+\Delta} \;=\; \begin{cases} |\mathbf{l}^2+\Delta|^{1/2}\;e^{\mathrm{i}\,\frac{\phi}{2}} \\ |\mathbf{l}^2+\Delta|^{1/2}\;e^{\mathrm{i}\,(\frac{\phi}{2}+\pi)} \end{cases} \tag{8.52}$$

with phases $\frac{\phi}{2} \in (\frac{\pi}{2}, \pi)$ and $\frac{\phi}{2} + \pi \in (\frac{3\pi}{2}, 2\pi)$, i.e. the poles are located in the second and the fourth quadrant of the complex plane. If we now consider the following contour



(8.53)

then Cauchy's residue theorem implies that the contour integral

$$\oint_\gamma \frac{1}{((-l^0)^2 + \mathbf{l}^2 + \Delta)^n} \, \mathrm{d}l^0 \;=\; 0 \tag{8.54}$$

vanishes because all poles are located in the red regions, hence no pole gets encircled by the contour γ. Sending the curved parts of this contour to infinity, their contributions to the integral become zero because the integrand falls off like $\sim \frac{1}{(l^0)^{2n}}$. Hence, we obtain the identity

$$\int_{-\infty}^{\infty} \frac{1}{((-l^0)^2 + \mathbf{l}^2 + \Delta)^n} \, \mathrm{d}l^0 \;+\; \int_{\mathrm{i}\infty}^{-\mathrm{i}\infty} \frac{1}{((-l^0)^2 + \mathbf{l}^2 + \Delta)^n} \, \mathrm{d}l^0 \;=\; 0 \quad , \tag{8.55}$$

which tells us that the integral over the real l^0 axis (which we would like to compute) can be computed by an integral over the imaginary l^0 axis. Introducing the new variable $l_E^0 := \mathrm{i}\, l^0$, we can write this equivalently as

$$\int_{-\infty}^{\infty} \frac{1}{((-l^0)^2 + \mathbf{l}^2 + \Delta)^n} \, \mathrm{d}l^0 \;=\; \mathrm{i} \int_{-\infty}^{\infty} \frac{1}{((l_E^0)^2 + \mathbf{l}^2 + \Delta)^n} \, \mathrm{d}l_E^0 \quad . \tag{8.56}$$

This reformulation of the integral is often called *Wick rotation* and the vector

$$l_E \;:=\; \begin{pmatrix} l_E^0 \\ \mathbf{l} \end{pmatrix} \;\in\; \mathbb{R}^d \tag{8.57}$$

is called the *Euclidean momentum*, which is motivated by the fact that the integrand depends on the Euclidean (in contrast to the Minkowski) norm square

$$l_E^2 \;:=\; (l_E^0)^2 + \mathbf{l}^2 \quad . \tag{8.58}$$

Summing up, we have shown that our master integral (8.46) can be computed equivalently in d-dimensional Euclidean space via

$$I_{d,n}(\Delta) \;=\; \int_{\mathbb{R}^d} \frac{1}{(l^2+\Delta)^n}\,\frac{\mathrm{d}l}{(2\pi)^d} \;=\; \mathrm{i}\int_{\mathbb{R}^d} \frac{1}{(l_E^2+\Delta)^n}\,\frac{\mathrm{d}l_E}{(2\pi)^d} \qquad (\text{for } \mathrm{Im}(\Delta)<0) \quad , \tag{8.59}$$

where $\mathrm{d}l_E := \mathrm{d}l_E^0\,\mathrm{d}l^1\,\cdots\,\mathrm{d}l^{d-1}$ denotes the d-dimensional Euclidean volume element.

Since the Wick-rotated integral (8.59) is invariant under d-dimensional rotations, it makes sense to work with d-dimensional spherical coordinates. This yields

$$\begin{aligned} I_{d,n}(\Delta) \;&=\; \mathrm{i}\int_{\mathbb{R}^d} \frac{1}{(l_E^2+\Delta)^n}\,\frac{\mathrm{d}l_E}{(2\pi)^d} \;=\; \frac{\mathrm{i}}{(2\pi)^d}\left(\int_{\mathbb{S}^{d-1}}\mathrm{d}\Omega_{d-1}\right)\int_0^\infty \frac{\rho^{d-1}}{(\rho^2+\Delta)^n}\,\mathrm{d}\rho \\ &=\; \frac{\mathrm{i}}{2\,(2\pi)^d}\left(\int_{\mathbb{S}^{d-1}}\mathrm{d}\Omega_{d-1}\right)\int_0^\infty \frac{(\rho^2)^{\frac{d-2}{2}}}{(\rho^2+\Delta)^n}\,\mathrm{d}(\rho^2) \quad , \end{aligned} \tag{8.60}$$

where $\rho=\sqrt{l_E^2}=\sqrt{(l_E^0)^2+\mathbf{l}^2}$ is the radius coordinate and $\mathrm{d}\Omega_{d-1}$ denotes the area element of the $d-1$-dimensional unit sphere $\mathbb{S}^{d-1}=\{x\in\mathbb{R}^d\,:\,x^2=1\}$.

Lemma 8.11. *The area of the $d-1$-dimensional unit sphere $\mathbb{S}^{d-1}$ is given by*

$$\int_{\mathbb{S}^{d-1}}\mathrm{d}\Omega_{d-1} \;=\; \frac{2\,\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \quad , \tag{8.61}$$

where

$$\Gamma(z) \;:=\; \int_0^\infty t^{z-1}\,e^{-t}\,\mathrm{d}t \tag{8.62}$$

denotes the gamma function.

Proof. This statement is proven by the following trick

$$\begin{aligned} (\sqrt{\pi})^d \;&=\; \left(\int_{-\infty}^\infty e^{-x^2}\,\mathrm{d}x\right)^d \;=\; \int_{\mathbb{R}^d} e^{-(x_1^2+x_2^2+\cdots+x_d^2)}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\cdots\,\mathrm{d}x_d \\ &=\; \left(\int_{\mathbb{S}^{d-1}}\mathrm{d}\Omega_{d-1}\right)\int_0^\infty e^{-r^2}\,r^{d-1}\,\mathrm{d}r \;=\; \left(\int_{\mathbb{S}^{d-1}}\mathrm{d}\Omega_{d-1}\right)\frac{1}{2}\int_0^\infty (r^2)^{\frac{d-2}{2}}e^{-r^2}\,\mathrm{d}(r^2) \\ &=\; \left(\int_{\mathbb{S}^{d-1}}\mathrm{d}\Omega_{d-1}\right)\frac{1}{2}\int_0^\infty t^{\frac{d}{2}-1}e^{-t}\,\mathrm{d}t \;=\; \left(\int_{\mathbb{S}^{d-1}}\mathrm{d}\Omega_{d-1}\right)\frac{1}{2}\,\Gamma(\tfrac{d}{2}) \quad , \end{aligned} \tag{8.63}$$

where in the second line we introduced d-dimensional spherical coordinates. □

Using this lemma, we continue our calculation in (8.60) and obtain

$$\begin{aligned} I_{d,n}(\Delta) \;&=\; \frac{\mathrm{i}}{2\,(2\pi)^d}\left(\int_{\mathbb{S}^{d-1}}\mathrm{d}\Omega_{d-1}\right)\int_0^\infty \frac{(\rho^2)^{\frac{d-2}{2}}}{(\rho^2+\Delta)^n}\,\mathrm{d}(\rho^2) \\ &=\; \frac{\mathrm{i}}{(4\pi)^{\frac{d}{2}}\,\Gamma(\frac{d}{2})}\int_0^\infty \frac{(\rho^2)^{\frac{d-2}{2}}}{(\rho^2+\Delta)^n}\,\mathrm{d}(\rho^2) \\ &=\; \frac{\mathrm{i}}{(4\pi)^{\frac{d}{2}}\,\Gamma(\frac{d}{2})}\,\frac{1}{\Delta^{n-\frac{d}{2}}}\int_0^1 t^{n-\frac{d}{2}-1}\,(1-t)^{\frac{d}{2}-1}\,\mathrm{d}t \quad , \end{aligned} \tag{8.64}$$

where in the last step we changed the integration variables according to $t = \Delta/(\rho^2 + \Delta)$. Let us observe that the remaining integral in (8.64) is related to the well-studied *Euler beta function*

$$B(z_1, z_2) := \int_0^1 t^{z_1 - 1} (1 - t)^{z_2 - 1} \quad , \tag{8.65a}$$

which itself is related to the gamma function by the identity

$$B(z_1, z_2) = \frac{\Gamma(z_1)\,\Gamma(z_2)}{\Gamma(z_1 + z_2)} \quad . \tag{8.65b}$$

This allows us to express (8.64) in the following very useful final form

$$I_{d,n}(\Delta) = \int_{\mathbb{R}^d} \frac{1}{(l^2 + \Delta)^n} \frac{\mathrm{d}l}{(2\pi)^d} = \frac{\mathrm{i}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \frac{1}{\Delta^{n - \frac{d}{2}}} \qquad (\text{for } \mathrm{Im}(\Delta) < 0) \quad , \tag{8.66}$$

which involves the well-studied gamma function.

Computing the loop integrals: With these preparations, we are now finally in the position to compute our loop integrals (8.40) and (8.50). For a reason that will become clear in a moment, we will keep the dimension d as a free parameter and investigate only later what happens when $d \to 4$ approaches the desired spacetime dimension. For the first loop integral (8.40), we find

$$\begin{aligned} \Delta_F(0) &= \lim_{\epsilon \to 0} \int_{\mathbb{R}^d} \frac{-\mathrm{i}}{l^2 + m^2 - \mathrm{i}\,\epsilon} \frac{\mathrm{d}l}{(2\pi)^d} = -\mathrm{i} \lim_{\epsilon \to 0} I_{d,1}(m^2 - \mathrm{i}\,\epsilon) \\ &= \lim_{\epsilon \to 0} \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1 - \frac{d}{2})}{\Gamma(1)} \frac{1}{(m^2 - \mathrm{i}\,\epsilon)^{1 - \frac{d}{2}}} = \frac{m^{d-2}}{(4\pi)^{\frac{d}{2}}} \Gamma(1 - \tfrac{d}{2}) \quad , \end{aligned} \tag{8.67}$$

where in the third step we used our general master integral formula (8.66), and in the last step we took the limit $\epsilon \to 0$ and used $\Gamma(1) = 0! = 1$. (Recall that $\Gamma(n) = (n-1)!$ for any positive integer n.) Similarly, we find for the second loop integral (8.50)

$$\begin{aligned} \mathrm{i}\,V(p^2) &= -\lim_{\epsilon \to 0} \int_0^1 \left(\int_{\mathbb{R}^d} \frac{1}{\left(l^2 + \alpha\,(1 - \alpha)\,p^2 + m^2 - \mathrm{i}\,\epsilon\right)^2} \frac{\mathrm{d}l}{(2\pi)^d} \right) \mathrm{d}\alpha \\ &= -\lim_{\epsilon \to 0} \int_0^1 I_{d,2}(\alpha\,(1 - \alpha)\,p^2 + m^2 - \mathrm{i}\,\epsilon)\ \mathrm{d}\alpha \\ &= \lim_{\epsilon \to 0} \int_0^1 \frac{-\mathrm{i}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2 - \frac{d}{2})}{\Gamma(2)} \frac{1}{\left(\alpha\,(1 - \alpha)\,p^2 + m^2 - \mathrm{i}\,\epsilon\right)^{2 - \frac{d}{2}}} \mathrm{d}\alpha \\ &= \int_0^1 \frac{-\mathrm{i}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(2 - \frac{d}{2})}{\left(\alpha\,(1 - \alpha)\,p^2 + m^2\right)^{2 - \frac{d}{2}}} \mathrm{d}\alpha \quad . \end{aligned} \tag{8.68}$$

From these expressions we can now finally identify the divergences of the loop diagrams. Recall that the gamma function $\Gamma(z)$ has poles at every non-positive integer $z = 0, -1, -2, \ldots$. If we

now try to insert $d = 4$ into (8.67) and (8.68), we obtain divergent expressions due to the poles $\Gamma(1 - \frac{d}{2}) = \Gamma(-1)$ and $\Gamma(2 - \frac{d}{2}) = \Gamma(0)$. The main trick of dimensional regularization is to consider a dimension $d = 4 - \varepsilon$ that is slightly smaller than our desired 4 spacetime dimensions, i.e. $\varepsilon > 0$ is very small, and characterize these divergences by studying a Laurent expansion of the relevant gamma functions around $\varepsilon = 0$. (The parameter ε is unrelated to the $-\mathrm{i}\,\epsilon$ term in the Feynman propagator, which is why it is denoted by a slightly different symbol. Luckily there is little chance of confusion because we have already carried out the limit $\epsilon \to 0$, so all $\varepsilon = 4 - d$ below are the ones associated with the spacetime dimension.) The Laurent expansions that are relevant for us can be looked up in the QFT literature (see e.g. Peskin/Schroeder (Appendix A.4)) and they read as follows

$$\Gamma(1 - \tfrac{d}{2}) \;=\; \Gamma(-1 + \tfrac{\varepsilon}{2}) \;=\; -\Big(\frac{2}{\varepsilon} + 1 - \gamma\Big) \;+\; \mathcal{O}(\varepsilon) \quad , \tag{8.69a}$$

$$\Gamma(2 - \tfrac{d}{2}) \;=\; \Gamma(\tfrac{\varepsilon}{2}) \;=\; \frac{2}{\varepsilon} - \gamma \;+\; \mathcal{O}(\varepsilon) \quad , \tag{8.69b}$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant. Using also a Taylor expansion of the form

$$A^{b\,\varepsilon} \;=\; e^{b\,\varepsilon\,\log(A)} \;=\; 1 + b\,\varepsilon\,\log(A) + \mathcal{O}(\varepsilon^2) \tag{8.70}$$

for m^{d-2} and $(4\pi)^{-\frac{d}{2}}$ in (8.67), and for $(4\pi)^{-\frac{d}{2}}$ and $\big(\alpha\,(1-\alpha)\,p^2 + m^2\big)^{\frac{d}{2}-2}$ in (8.68), we obtain the Laurent expansions

$$\Delta_F(0) \;=\; -\frac{m^2}{(4\pi)^2}\,\Big(\frac{2}{\varepsilon} + 1 - \gamma + \log(4\pi) - \log\big(m^2\big)\Big) \;+\; \mathcal{O}(\varepsilon) \quad . \tag{8.71}$$

and

$$\mathrm{i}\,V(p^2) \;=\; \frac{-\mathrm{i}}{(4\pi)^2}\int_0^1 \Big(\frac{2}{\varepsilon} - \gamma + \log(4\pi) - \log\Big(\alpha\,(1-\alpha)\,p^2 + m^2\Big)\Big)\,\mathrm{d}\alpha \;+\; \mathcal{O}(\varepsilon) \quad . \tag{8.72}$$

As a consequence of the $\frac{1}{\varepsilon}$-poles, these loop integrals are divergent when sending $\varepsilon \to 0$, which was expected from our heuristic analysis of their superficial degree of divergence.

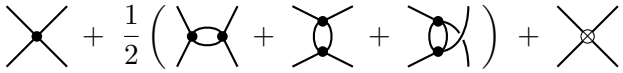
Conclusions: Inserting (8.71) and (8.72) into our formulas for the counterterms (8.39) and (8.45), we obtain

$$\delta_Z = 0 + \mathcal{O}(\lambda^2) + \mathcal{O}(\varepsilon) \quad , \tag{8.73a}$$

$$\delta_m = \frac{\lambda\, m^2}{32\pi^2} \left(\frac{2}{\varepsilon} + 1 - \gamma + \log(4\pi) - \log\left(m^2\right) \right) + \mathcal{O}(\lambda^2) + \mathcal{O}(\varepsilon) \quad , \tag{8.73b}$$

$$\delta_\lambda = \frac{\lambda^2}{32\pi^2} \int_0^1 \left(\frac{6}{\varepsilon} - 3\,\gamma + 3\,\log(4\pi) - \log\left(m^2 - \alpha\,(1-\alpha)\,4m^2\right) - 2\,\log\left(m^2\right) \right) \mathrm{d}\alpha + \mathcal{O}(\lambda^3) + \mathcal{O}(\varepsilon) \quad . \tag{8.73c}$$

As expected, the counterterms are divergent (in the limit $\varepsilon \to 0$), which is however not a problem since, by design, they should cancel the divergences appearing in loop diagrams such that the time-ordered n-point functions and scattering amplitudes are finite. To see an explicit example of such cancellations, let us consider again the $2 \to 2$ scattering amplitude

$\langle q_1, q_2; \text{out} | k_1, k_2; \text{in} \rangle =$  $+ \mathcal{O}(\lambda^3)$ (8.74)

given explicitly in (8.44). Inserting our results for the loop integral (8.72) and the counterterm (8.73) into this expression, we find

$$\mathcal{A}(k_1, k_2, q_1, q_2) = -\mathrm{i}\,\lambda - \frac{\mathrm{i}\,\lambda^2}{32\pi^2} \int_0^1 \left(\log\left(\frac{m^2 + \alpha\,(1-\alpha)\,s}{m^2 - \alpha\,(1-\alpha)\,4m^2} \right) + \log\left(\frac{m^2 + \alpha\,(1-\alpha)\,t}{m^2} \right) + \log\left(\frac{m^2 + \alpha\,(1-\alpha)\,u}{m^2} \right) \right) \mathrm{d}\alpha + \mathcal{O}(\lambda^3) + \mathcal{O}(\varepsilon) \quad , \tag{8.75}$$

where we recall the Mandelstam variables $s = (k_1 + k_2)^2$, $t = (k_1 - q_1)^2$ and $u = (k_1 - q_2)^2$. Note that this scattering amplitude does not contain any $\frac{1}{\varepsilon}$-poles, which means that we can remove the regulator by sending $\varepsilon \to 0$. The underlying mechanism is that the $\frac{1}{\varepsilon}$-poles in the loop diagrams are canceled precisely by the poles in the counterterm (8.73), leading to a finite result. It is important to emphasize that the scattering amplitude is *finite for all momenta* k_1, k_2, q_1, q_2 and not only at the particular energy scale at which we have introduced the renormalization condition (8.32). With more sophisticated techniques, one can show that such cancellations of divergences happen in all time-ordered n-point functions and all scattering amplitudes, i.e. the renormalization process successfully cured the ultraviolet divergences of our QFT.

Further reading

For more details about renormalization, see e.g. Peskin/Schroeder (Chapter 10), Nastase (Chapters 32–35) and Srednicki (Chapters 14–18) from our reading list in Section 1.3.

Chapter 9

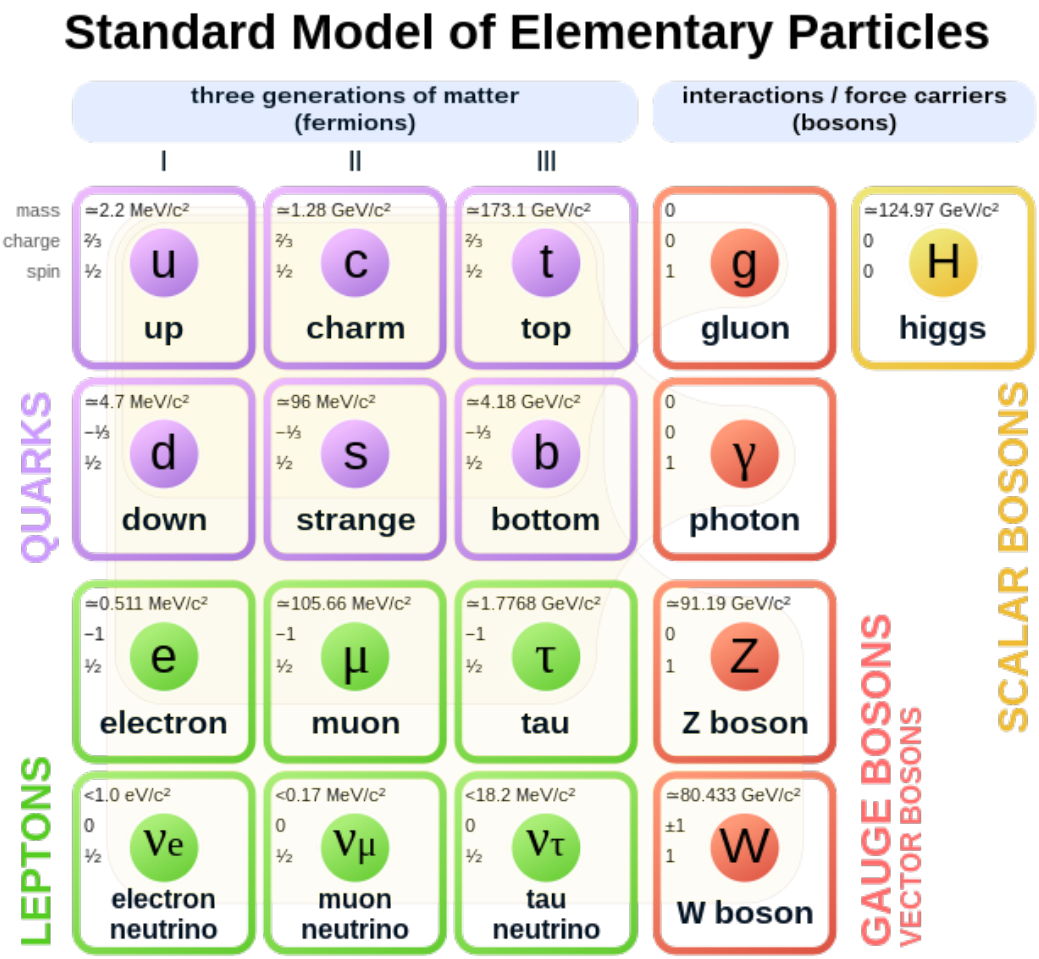
Standard model of particle physics

This chapter introduces the building blocks that are required to understand the structure of the standard model of particle physics. It culminates in a detailed description of the relevant classical action functional that underlies this QFT. Further aspects of the standard model will be discussed in some of the student projects/presentations.

9.1 Brief overview

The standard model of particle physics is an impressive construct. It is a QFT that describes, very successfully, all observed fundamental particles in nature (electrons, neutrinos, quarks, ...) and three of the four fundamental interactions (electromagnetic, weak and strong interaction). Unfortunately, the standard model does not include the gravitational interaction, which according to our current understanding can not be treated consistently within the framework of QFT. The issue is that the quantization of gravity leads to a non-renormalizable QFT (in the sense of Definition 8.2), hence our renormalization methods from Chapter 8 can not be applied to yield a predictive QFT for gravity. In contrast to that, the standard model of particle physics is a renormalizable QFT in $d = 4$ spacetime dimensions. During the past few decades there has been plenty of research towards constructing a theory of quantum gravity that resolves these issues, leading to viable candidates such as string theory and loop quantum gravity. However, a full understanding of quantum gravity is still elusive. It is important to stress that the problem of quantum gravity is more of a conceptual than practical nature: At the typical energy scales that are relevant for particle physics, say a few teraelectron volts TeV as at the Large Hadron Collider (LHC), the gravitational interaction is extremely weak so that gravitational phenomena can usually be neglected. So a theory of quantum gravity is not necessarily needed for describing particle physics phenomena, which is the main reason why the standard model of particle physics (which excludes gravity) works so well in practice. However, in more extreme scenarios, such as in the early universe or in the vicinity of black holes, quantum gravity effects will become relevant so that an extension of the standard model to include gravity will eventually be required.

Explaining the standard model of particle physics is not so easy because, being a theory for *all* known fundamental particles and *all* their non-gravitational interactions, it is necessarily quite big. I think the best place to start is to look at the following schematic description (taken from the Wikipedia website) that nicely summarizes the particle content of the standard model:



Let us look more closely at this picture and extract some relevant information:

- The fourth column lists the so-called *gauge bosons* of the standard model. We recognize an old friend, the *photon* γ, which, as we have seen in Chapter 7, is the particle responsible for mediating the electromagnetic interaction. The other gauge bosons play a similar role: The *W^+, W^- and Z bosons* mediate the weak interaction and the *gluons* g mediate the strong interaction. Hence, they are suitable generalizations (to be made precise in this chapter) of the electromagnetic potential/photon.

- The first three columns list the *matter fermions* that are the building blocks for the matter observed in nature. Let us start by looking at the first column. We again recognize an old friend, the *electron* e, which comes with an associated *electron neutrino* ν_e; you probably have heard before about neutrinos in the context of radioactive decays. The electron and its neutrino are among the so-called *leptons*, which are particles that do not participate in the strong interaction. The other two matter particles in the first column are the *up quark* u and the *down quark* d. The word *quark* is used to describe particles that participate also in the strong interaction, in contrast to the leptons which do not. Quarks are the building blocks for composite (i.e. non-fundamental) particles called *baryons* and *mesons*, e.g. the proton is a

bound state *uud* of two up and one down quark, and the neutron is a bound state *udd* of one up and two down quarks. Note that this means that we can build all atoms (i.e. the nuclei and their electron clouds) by using fundamental particles from the first column.

- The particles in the second and third column are copies of the ones in the first column, which have the same properties but different (heavier) mass. For example, the *muon* μ is a heavier cousin of the electron e and the *tau* τ is an even heavier cousin. These three versions of the matter fermions (that are only distinguished by their masses) are often called the *three generations*.

- Last but not least there is the fifth column, which contains the *Higgs boson*. This ingredient is somewhat special, because the Higgs is the only particle of spin 0 in the standard model, while all matter fermions have spin $\frac{1}{2}$ and all gauge bosons have spin 1. The role of the Higgs is to give mass to the matter fermions, to some of the gauge bosons (namely the W^+, W^- and Z bosons associated with the weak interaction) and also to itself through a mechanism that is called *spontaneous symmetry breaking* or the *Higgs mechanism*.

The aim of this chapter is to explain the mathematics behind these individual building blocks of the standard model as well as their interplay. We will start with the gauge bosons, i.e. the fourth column, which are the mediators of the electromagnetic, weak and strong interactions. They can be described by a non-trivial generalization of the electromagnetic potential and its gauge symmetries from Chapters 6 and 7, which are called (non-Abelian) *Yang-Mills theories*. The input for such theories is a choice of *Lie group* G that dictates the number of components of the gauge field, as well as their dynamics, and the structure of the gauge symmetries. For the Abelian group $G = \mathsf{U}(1) = \{U \in \mathbb{C} : U^* U = 1\}$, Yang-Mills theory specializes to electromagnetism, hence it is indeed a generalization of the latter. The Lie group of the standard model of particle physics is the product $G = \mathsf{U}(1) \times \mathsf{SU}(2) \times \mathsf{SU}(3)$ of the circle group $\mathsf{U}(1)$ with the two-dimensional and the three-dimensional special unitary group. As we will understand at the end of this module, the factor $\mathsf{U}(1) \times \mathsf{SU}(2)$ gives rise to the photon and the weak gauge bosons (these are $\dim\big(\mathsf{U}(1) \times \mathsf{SU}(2)\big) = 4$ many, namely γ, W^+, W^- and Z), while the factor $\mathsf{SU}(3)$ gives rise to the $\dim \mathsf{SU}(3) = 8$ many gluons g.

Each generation of quarks and leptons in the first three columns is described by a certain family of Dirac fields in the sense of Chapter 5 that is suitably coupled to the gauge bosons via a generalization of the minimal gauge coupling construction from electromagnetism, see Section 7.1. We will later see that, in the context of Yang-Mills theory with Lie group G, such coupling is determined by choosing a linear *representation* of G on a vector space V. The punchline will be that the coupling between the gauge fields and a Dirac field $\Psi : \mathbb{R}^d \to \mathbb{C}^N \otimes V\,,\ x \mapsto \Psi(x)$ taking values in a representation V is governed by a suitable concept of covariant derivative. While the above captures the essence how a coupling between gauge fields and matter fermions can be established, the standard model comes with a small but physically important tweak: Since the standard model is defined in $d = 4$ spacetime dimensions, one can decompose a Dirac field $\Psi = (\Psi_L, \Psi_R)$ into its left and right-handed chiral components, see the explanation around Eqn. (5.33). What happens in nature is that Ψ_L and Ψ_R participate differently in the interactions, which means that they take values in different representations of G, say $\Psi_L(x) \in \mathbb{C}^2 \otimes V_L$ and $\Psi_R(x) \in \mathbb{C}^2 \otimes V_R$. So the standard model must be formulated using the finer concept of left and right-handed Weyl spinor fields, in contrast to Dirac fields that consist of both a left and a right-handed Weyl field. In short,

people often say that the standard model is a *chiral gauge theory* as it treats left and right-handed chiralities differently.

The Higgs boson is described by a complex scalar field $\Phi : \mathbb{R}^d \to \mathbb{C} \otimes V_H\,,\ x \mapsto \Phi(x)$ that couples to the gauge bosons through a suitable representation V_H of the Lie group G. It also couples to the matter fermions through a so-called *Yukawa interaction term* in the action. These two kinds of couplings are crucial because, after spontaneous symmetry breaking, they will generate mass terms for the matter fermions and also for the W^+, W^- and Z bosons.

You can now rightfully ask a lot of questions: How can someone come up with such a sophisticated and complicated model? Why is $G = \mathsf{U}(1) \times \mathsf{SU}(2) \times \mathsf{SU}(3)$ the correct choice of Lie group? How do I choose the correct representations V of G? Why are there three generations of matter, and not only 1 or as many as 42? These are indeed good questions that people have thought about, but I think that nobody has a good answer. Our current mathematical description of the standard model requires us to make certain choices along the way, such as the Lie group G, its representations, the number of generations and some numerical parameters in the action functional, all of which require experimental input. In fact, the development of the standard model of particle physics took various decades, and it was due to a constant back-and-forth between experiment and theory that such a successful model for fundamental physics could be developed.

9.2 Lie groups, Lie algebras and representations

As it becomes evident from our informal discussion in the previous section, the concepts of Lie groups, Lie algebras and their representations are crucial to understand the structure of the standard model of particle physics. The aim of this section is to give a very brief (and hence necessarily incomplete) introduction to these rich topics, focusing mostly on the specific examples that we require to set up the standard model.

Informally speaking, Lie groups are used to describe continuous families of transformations that depend *smoothly* on their parameters. The general mathematical definition combines concepts from differential geometry and group theory, and it reads as follows:

Definition 9.1 (Lie group)**.** A *Lie group* is a smooth manifold G that is equipped with a smooth map $G \times G \to G\,,\ (g', g) \mapsto g'\,g$ (called group multiplication), an element $e \in G$ (called identity element) and a smooth map $G \to G\,,\ g \mapsto g^{-1}$ (called inversion). These structures have to satisfy the axioms of a group, i.e.

(i) *Associativity:* $(g''\,g')\,g = g''\,(g'\,g)$ for all $g, g', g'' \in G$

(ii) *Identity property:* $g\,e = g = e\,g$ for all $g \in G$

(iii) *Inverse property:* $g^{-1}\,g = e = g\,g^{-1}$ for all $g \in G$

A Lie group G is called *Abelian* if the group multiplication is commutative, i.e. $g'\,g = g\,g'$ for all $g, g' \in G$, and otherwise it is called *non-Abelian*.

This definition is very general and hence it captures many relevant examples. For instance, one can endow the (proper and orthochronous) Lorentz group $\mathsf{SO}_0(d-1,1)$, the spin group $\mathsf{Spin}(d-1,1)$ and the Poincaré group with a canonical manifold structure such that they become Lie groups. In

the context of the standard model of particle physics, we do not necessarily need this general definition of a Lie group and it suffices to understand the following families of examples.

Example 9.2 (The Lie groups $\mathsf{U}(n)$ and $\mathsf{SU}(n)$)**.** Let $n \geq 1$ be a positive integer. The *unitary group of degree* n is defined as the submanifold

$$\mathsf{U}(n) := \left\{ U \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \,:\, U^\dagger U = \mathbf{1} = U\, U^\dagger \right\} \subseteq \mathrm{Mat}_{n\times n}(\mathbb{C}) \tag{9.1}$$

that consists of all unitary $n \times n$-matrices, together with the group multiplication $U'U$ given by matrix multiplication, the identity element $e = \mathbf{1}$ given by the identity matrix and the inversion U^{-1} given by the inverse matrix. (Note that the latter is equal to the adjoint matrix $U^{-1} = U^\dagger$ because we consider unitary matrices.) Setting $n = 1$, we observe that the circle group $\mathsf{U}(1) = \{U \in \mathbb{C} \,:\, U^* U = 1\} \subseteq \mathbb{C}$ is the unitary group of degree 1. The *special unitary group of degree* n is defined as the Lie subgroup

$$\mathsf{SU}(n) := \left\{ U \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \,:\, U^\dagger U = \mathbf{1} = U\, U^\dagger \;\text{ and }\; \det U = 1 \right\} \subseteq \mathsf{U}(n) \tag{9.2}$$

that consists of all unitary $n \times n$-matrices with determinant 1. Since the multiplication of matrices of size $n \geq 2$ is non-commutative, we observe that $\mathsf{U}(n)$ and $\mathsf{SU}(n)$ are non-Abelian Lie groups for $n \geq 2$. In contrast to this, circle group $\mathsf{U}(1)$ is an Abelian Lie group.

Since a Lie group G describes by design a notion of smooth families of transformations, one should expect that there is an associated notion of infinitesimal transformations that can be obtained via a suitable differentiation procedure. This is indeed the case and can be formalized using the concept of tangent spaces from differential geometry. More precisely, the infinitesimal transformations are characterized by the tangent space $\mathfrak{g} := T_e G$ at the identity element $e \in G$, which can be endowed with a Lie algebra structure $[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is induced by the Lie group structure on G. We have already seen some examples of Lie algebras in this module, but we have never written down a general definition. So let us do this now.

Definition 9.3 (Lie algebra)**.** A (real) *Lie algebra* is a real vector space $\mathfrak{g}$ together with a bilinear map $[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called Lie bracket) that satisfies the following axioms:

(i) *Antisymmetry:* $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$

(ii) *Jacobi identity:* $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{g}$

A Lie algebra $\mathfrak{g}$ is called *Abelian* if the Lie bracket is trivial, i.e. $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$, and otherwise it is called *non-Abelian*.

Let us illustrate this concept by working out the Lie algebras associated with the unitary and special unitary groups.

Example 9.4 (The Lie algebras $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$)**.** Consider the unitary Lie group $\mathsf{U}(n)$ from Example 9.2, whose elements are unitary $n \times n$-matrices U. In the vicinity of the identity element

$\mathbf{1} \in \mathsf{U}(n)$, we can write such matrices in exponential form $U = e^X$, where X is some small $n \times n$-matrix. A first-order Taylor expansion in X of the unitarity condition $U^\dagger U = \mathbf{1}$ yields

$$U^\dagger U = e^{X^\dagger}\, e^X = \mathbf{1} + X^\dagger + X + \mathcal{O}(X^2) \stackrel{!}{=} \mathbf{1} \quad , \tag{9.3}$$

which means that the exponent must be an anti-Hermitian matrix $X^\dagger = -X$. (Note that the second condition $U\,U^\dagger = \mathbf{1}$ yields the same anti-Hermiticity condition for X.) This implies that the Lie algebra of $\mathsf{U}(n)$ is given by the vector space

$$\mathfrak{u}(n) := \Big\{ X \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \,:\, X^\dagger = -X \Big\} \tag{9.4a}$$

of anti-Hermitian $n \times n$-matrices, on which the canonically defined Lie bracket is given by the matrix commutator

$$[\,\cdot\,,\,\cdot\,] \,:\, \mathfrak{u}(n) \times \mathfrak{u}(n) \longrightarrow \mathfrak{u}(n) \;,\quad (X,Y) \longmapsto [X,Y] := X\,Y - Y\,X \quad . \tag{9.4b}$$

Note that $[X,Y] \in \mathfrak{u}(n)$ is indeed anti-Hermitian for all $X, Y \in \mathfrak{u}(n)$, which is shown by a short calculation

$$[X,Y]^\dagger = [Y^\dagger, X^\dagger] = [Y,X] = -[X,Y] \quad . \tag{9.5}$$

For the special unitary Lie group $\mathsf{SU}(n) \subseteq \mathsf{U}(n)$ there is the extra condition that $\det U = 1$ must be 1. Using the relationship $\det(e^X) = e^{\mathrm{Tr}(X)}$ between determinant and trace, we find that the Lie algebra of $\mathsf{SU}(n)$ is given by the vector space

$$\mathfrak{su}(n) := \Big\{ X \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \,:\, X^\dagger = -X \;\text{ and }\; \mathrm{Tr}(X) = 0 \Big\} \tag{9.6a}$$

of anti-Hermitian and trace-free $n \times n$-matrices, on which the canonically defined Lie bracket is given again by the matrix commutator

$$[\,\cdot\,,\,\cdot\,] \,:\, \mathfrak{su}(n) \times \mathfrak{su}(n) \longrightarrow \mathfrak{su}(n) \;,\quad (X,Y) \longmapsto [X,Y] := X\,Y - Y\,X \quad . \tag{9.6b}$$

Note that $[X,Y] \in \mathfrak{su}(n)$ is indeed trace-free for all $X, Y \in \mathfrak{su}(n)$, which is shown by using the cyclicity property of the trace

$$\mathrm{Tr}([X,Y]) = \mathrm{Tr}(X\,Y) - \mathrm{Tr}(Y\,X) = \mathrm{Tr}(X\,Y) - \mathrm{Tr}(X\,Y) = 0 \quad . \tag{9.7}$$

We observe that $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$ are non-Abelian Lie algebras for $n \geq 2$, and that the Lie algebra $\mathrm{u}(1) = \mathrm{i}\,\mathbb{R}$ of the circle group is Abelian and 1-dimensional.

Remark 9.5 (Structure constants of a Lie algebra)**.** Given any Lie algebra $\mathfrak{g}$, we can pick a basis $\{X_a \in \mathfrak{g}\}$ of its underlying vector space. Since the Lie bracket is bilinear, it is completely characterized by its values $[X_a, X_b] \in \mathfrak{g}$ on the basis elements, which can be expanded in the chosen basis as

$$[X_a, X_b] = f_{ab}^{\,c}\, X_c \quad . \tag{9.8}$$

The expansion coefficients $f_{ab}^c \in \mathbb{R}$ are called the *structure constants* of the Lie algebra in the chosen basis. At the level of the structure constants, antisymmetry of the Lie bracket reads as $f_{ab}^c = -f_{ba}^c$ and the Jacobi identity translates to

$$f_{ad}^e\, f_{bc}^d + f_{bd}^e\, f_{ca}^d + f_{cd}^e\, f_{ab}^d = 0 \quad . \tag{9.9}$$

This description of Lie algebras in terms of a basis is what you'll typically see in physics textbooks. Since using a basis X_a introduces further indices and thereby complicates notation, we will present most of our constructions in this chapter in a basis independent way.

To get a better feeling for how picking a basis works in practice, let us consider the Lie algebra $\mathfrak{su}(2)$ of anti-Hermitian and trace-free 2×2-matrices. A basis for $\mathfrak{su}(2)$ is given by the following matrices

$$X_1 = -\frac{\mathrm{i}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad X_2 = -\frac{\mathrm{i}}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad , \quad X_3 = -\frac{\mathrm{i}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \tag{9.10}$$

which you probably recognize as the Pauli matrices rescaled by a factor of $-\frac{\mathrm{i}}{2}$. Computing the commutators, one finds

$$[X_a, X_b] = \epsilon_{abc}\, X_c \quad , \tag{9.11}$$

i.e. the structure constants $f_{ab}^c = \epsilon_{abc}$ are given by the epsilon-tensor. The reason why there is no factor i on the right-hand side, as you would expect by comparing this with the $\mathfrak{su}(2)$ Lie algebra $[L_a, L_b] = \mathrm{i}\,\epsilon_{abc}\, L_c$ from your quantum mechanics module, is that we have taken different conventions for exponentials. Our convention was to write $U = e^X$, leading to anti-Hermitian exponents X, while in quantum mechanics people typically work with $U = e^{-\mathrm{i}\,L}$, leading to Hermitian exponents L. Note that one can pass from one convention to the other by setting $L_a = \mathrm{i}\, X_a$, which takes care of the factor i in the Lie bracket relations.

By design, a Lie group G describes an abstract notion of transformations and its Lie algebra $\mathfrak{g}$ describes an abstract notion of infinitesimal transformations. To let these transformations act concretely on other objects, such as a vector space V, one introduces the concept of representations. To understand the following definition, we observe that associated with every vector space V is the Lie group $\mathsf{GL}(V) \subseteq \mathrm{Lin}(V, V)$ of invertible linear maps $L : V \to V$ from V to itself, with group multiplication $L' \circ L$ given by composition of linear maps, identity element id given by the identity map and inversion L^{-1} given by the inverse linear map.

Definition 9.6 (Representation of a Lie group)**.** A *(linear) representation* of a Lie group G on a real or complex vector space V is a smooth group homomorphism

$$\rho \,:\, G \longrightarrow \mathsf{GL}(V) \;, \quad g \longmapsto \rho(g) \quad , \tag{9.12}$$

i.e. it preserves the group multiplication $\rho(g'\, g) = \rho(g') \circ \rho(g)$, for all $g, g' \in G$. (From this property one can deduce that the identity element $\rho(e) = \mathrm{id}$ and inverses $\rho(g^{-1}) = \big(\rho(g)\big)^{-1}$, for all $g \in G$, are preserved too.) In the case where V is a complex vector space endowed with a sesquilinear inner product $\langle\, \cdot\,, \cdot\,\rangle : V \times V \to \mathbb{C}$, one says that a representation ρ is *unitary* if it preserves the inner product in the sense that

$$\big\langle \rho(g)(v), \rho(g)(v') \big\rangle \;=\; \langle v, v' \rangle \quad , \tag{9.13}$$

for all $g \in G$ and $v, v' \in V$. The notation $\rho(g)(v) \in V$ means the application of the linear map $\rho(g) : V \to V$ on the element $v \in V$.

The classification of linear representations of Lie groups is an interesting and non-trivial problem that is studied in the mathematical discipline called *representation theory*. Luckily, for understanding the standard model of particle physics, it suffices to know only some very specific representations of the special unitary group $\mathsf{SU}(n)$ and the representations of the circle group $\mathsf{U}(1)$.

Example 9.7 (Important representations of $\mathsf{SU}(n)$)**.** We list some basic representations of $\mathsf{SU}(n)$ that are needed for constructing the standard model of particle physics.

1. *Trivial representation:* The trivial representation on the 1-dimensional complex vector space $\mathbb{C}$ is given by the smooth group homomorphism

$$\rho_{\text{triv}} \,:\, \mathsf{SU}(n) \longrightarrow \mathsf{GL}(\mathbb{C}) \;,\quad U \longmapsto \rho_{\text{triv}}(U) = \text{id} \tag{9.14}$$

   that sends every $U \in \mathsf{SU}(n)$ to the identity map. This representation is unitary with respect to the standard inner product $\langle v, v' \rangle = v^* \, v'$ on $\mathbb{C}$.

2. *Defining representation:* Consider the standard n-dimensional complex vector space $\mathbb{C}^n$. Recall from linear algebra that $\text{Lin}(\mathbb{C}^n, \mathbb{C}^n) \cong \text{Mat}_{n\times n}(\mathbb{C})$ can be identified with the $n \times n$-matrices with complex entries, hence $\mathsf{GL}(\mathbb{C}^n)$ can be identified with the Lie group of invertible $n \times n$-matrices. Explicitly, each $n \times n$-matrix $A \in \text{Mat}_{n\times n}(\mathbb{C})$ defines via matrix multiplication a linear map $\mathbb{C}^n \to \mathbb{C}^n \,,\; v \mapsto A\, v$, and every linear map can be written uniquely in this way. The defining representation

$$\rho_{\text{def}} \,:\, \mathsf{SU}(n) \longrightarrow \mathsf{GL}(\mathbb{C}^n) \;,\quad U \longmapsto \rho_{\text{def}}(U) = U \tag{9.15}$$

   is then simply given by the embedding of the unitary matrices with determinant 1 into all invertible matrices. Endowing $\mathbb{C}^n$ with its standard inner product $\langle v, v' \rangle := v^\dagger \, v'$, one easily checks that this representation is unitary

$$\big\langle \rho_{\text{def}}(U)(v), \rho_{\text{def}}(U)(v') \big\rangle = \langle U\, v, U\, v' \rangle = v^\dagger \, U^\dagger \, U \, v' = v^\dagger \, v' = \langle v, v' \rangle \quad . \tag{9.16}$$

3. *Adjoint representation:* Every Lie group G has a canonical linear representation on its Lie algebra $\mathfrak{g} := T_e G$, which is called the *adjoint representation*. For $\mathsf{SU}(n)$, the adjoint representation

$$\rho_{\text{ad}} \,:\, \mathsf{SU}(n) \longrightarrow \mathsf{GL}(\mathfrak{su}(n)) \tag{9.17a}$$

   is given element-wise in terms of matrix multiplication by

$$\rho_{\text{ad}}(U)(X) \,:=\, U\, X\, U^{-1} \,=\, U\, X\, U^\dagger \quad , \tag{9.17b}$$

where in the second step we used that U is unitary, i.e. $U^{-1} = U^\dagger$. This is indeed well-defined: One easily checks that the matrix $\rho_{\mathrm{ad}}(U)(X)$ is anti-Hermitian

$$\big(\rho_{\mathrm{ad}}(U)(X)\big)^\dagger = \big(U\,X\,U^\dagger\big)^\dagger = U\,X^\dagger\,U^\dagger = -\rho_{\mathrm{ad}}(U)(X) \tag{9.18}$$

and trace-free

$$\mathrm{Tr}\big(\rho_{\mathrm{ad}}(U)(X)\big) = \mathrm{Tr}\big(U\,X\,U^\dagger\big) = \mathrm{Tr}\big(U^\dagger\,U\,X\big) = \mathrm{Tr}(X) = 0 \quad , \tag{9.19}$$

hence $\rho_{\mathrm{ad}}(U)(X) \in \mathfrak{su}(n)$ defines an element of the Lie algebra. One also checks easily that ρ_{ad} defines a representation, which follows from the short calculation

$$\rho_{\mathrm{ad}}(U'\,U)(X) = (U'\,U)\,X\,(U'\,U)^\dagger = U'\,\big(U\,X\,U^\dagger\big)\,U'^\dagger = \rho_{\mathrm{ad}}(U')\big(\rho_{\mathrm{ad}}(U)(X)\big) \quad . \tag{9.20}$$

Note that the representations above also make sense when we replace $\mathsf{SU}(n)$ by the unitary Lie group $\mathsf{U}(n)$.

Example 9.8 (Representations of $\mathsf{U}(1)$)**.** It is quite easy to write down explicit examples of unitary linear representations of the circle group $\mathsf{U}(1) = \{U \in \mathbb{C} \,:\, U^*\,U = 1\} \subseteq \mathbb{C}$ on the 1-dimensional complex vector space $\mathbb{C}$. In this case we have that $\mathsf{GL}(\mathbb{C}) = \{U \in \mathbb{C} \,:\, U \neq 0\} = \mathbb{C} \setminus \{0\}$ is the complex plane minus the origin. For each integer $k \in \mathbb{Z}$, we can define a unitary representation

$$\rho^{(k)} \,:\, \mathsf{U}(1) \longrightarrow \mathsf{GL}(\mathbb{C}) \;,\quad U \longmapsto U^k \tag{9.21}$$

by taking U to the power k. Indeed, one easily checks that $\rho^{(k)}(U'\,U) = (U'\,U)^k = U'^k\,U^k = \rho^{(k)}(U')\,\rho^{(k)}(U)$, for all $U, U' \in \mathsf{U}(1)$. With some more efforts, which won't be discuss here, one can show that these are all irreducible linear representations of $\mathsf{U}(1)$, i.e. every other linear representation can be build from these $\rho^{(k)}$'s.

Remark 9.9 (New representations from old ones)**.** There exist general techniques to construct new representations out of given ones. A detailed discussion of these methods goes beyond the scope of this module, but as a taster I would like to sketch some of the basic ideas.

(i) *Tensor product representations:* Let G be a Lie group and suppose that we have given two vector spaces V and W with representations ρ_V and ρ_W, respectively. Then the tensor product representation is defined by

$$\rho_{V\otimes W} \,:\, G \longrightarrow \mathsf{GL}(V \otimes W) \;,\quad g \longmapsto \rho_V(g) \otimes \rho_W(g) \quad . \tag{9.22}$$

You probably have seen simple examples of such tensor product representations when you've studied spin in quantum mechanics, which are unitary $\mathsf{SU}(2)$ representations. More concretely, a non-relativistic spin $\frac{1}{2}$ particle is described by the defining representation of $\mathsf{SU}(2)$, which in physics is typically written in terms of the states $|\uparrow\rangle, |\downarrow\rangle \in \mathbb{C}^2$. The tensor product of this representation with itself is described by the tensor product states $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$. A standard exercise in quantum mechanics is then to show that this tensor product representation can be decomposed into a spin 1 and a spin 0 representation by using Clebsch-Gordan coefficients.

(ii) *Dual representations:* Let G be a Lie group and suppose that we have given a representation ρ on a vector space V. Recall that the dual vector space $V^* := \mathrm{Lin}(V, \mathbb{K})$ is given by the vector space of linear maps $f : V \to \mathbb{K}$ from V to the 1-dimensional vector space $\mathbb{K}$, i.e. $\mathbb{K} = \mathbb{R}$ in the real case and $\mathbb{K} = \mathbb{C}$ in the complex case. Then the dual representation

$$\rho^* \,:\, G \longrightarrow \mathsf{GL}(V^*) \tag{9.23a}$$

is defined element-wise by

$$\rho^*(g)(f) \,:=\, f \circ \rho(g^{-1}) \quad , \tag{9.23b}$$

for all $g \in G$ and $f \in V^*$.

There is also a concept of representations for Lie algebras on vector spaces. To understand the following definition, we observe that associated with every vector space V is the Lie algebra $\mathfrak{gl}(V) := \mathrm{Lin}(V, V)$ of linear maps $L : V \to V$ from V to itself, with Lie bracket $[L, L'] := L \circ L' - L' \circ L$ given by the commutator.

Definition 9.10 (Representation of a Lie algebra)**.** A *representation* of a Lie algebra $\mathfrak{g}$ on a real or complex vector space V is a Lie algebra homomorphism

$$\underline{\rho} \,:\, \mathfrak{g} \longrightarrow \mathfrak{gl}(V) \;, \quad X \longmapsto \underline{\rho}(X) \quad , \tag{9.24}$$

i.e. a linear map that preserves the Lie bracket

$$\underline{\rho}\big([X, Y]\big) = \big[\underline{\rho}(X), \underline{\rho}(Y)\big] \quad , \tag{9.25}$$

for all $X, Y \in \mathfrak{g}$. In the case where V is a complex vector space endowed with a sesquilinear inner product $\langle \,\cdot\,, \cdot\, \rangle : V \times V \to \mathbb{C}$, one says that a representation $\underline{\rho}$ is *unitary* if it preserves the inner product in the sense that

$$\big\langle \underline{\rho}(X)(v), v' \big\rangle + \big\langle v, \underline{\rho}(X)(v') \big\rangle \,=\, 0 \quad , \tag{9.26}$$

for all $X \in \mathfrak{g}$ and $v, v' \in V$. The notation $\underline{\rho}(X)(v) \in V$ means the application of the linear map $\underline{\rho}(X) : V \to V$ on the element $v \in V$.

One can show that every linear representation ρ of a Lie group G on a vector space V induces a representation $\underline{\rho}$ of its underlying Lie algebra $\mathfrak{g} = T_e G$ on the same V. Instead of explaining this construction in full generality, we give some relevant examples.

Example 9.11 (Important representations of $\mathfrak{su}(n)$)**.** Before going into the examples, let me sketch the basic idea how one can construct Lie algebra representations from linear Lie group representations: Given any linear representation $\rho : \mathsf{SU}(n) \to \mathsf{GL}(V)$ of the Lie group $\mathsf{SU}(n)$, we can consider the linear map $\rho(e^X) : V \to V$ that is associated to $U = e^X \in \mathsf{SU}(n)$ for a Lie algebra element $X \in \mathfrak{su}(n)$. Using a first-order Taylor expansion in X, we define

$$\rho(e^X) =: \mathrm{id} + \underline{\rho}(X) + \mathcal{O}(X^2) \quad . \tag{9.27}$$

From the Baker-Campbell-Hausdorff formula $e^X\, e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\cdots}$ and the property that ρ is a Lie group representation, one shows that $\underline{\rho}$ is a Lie algebra representation, i.e.

$$\underline{\rho}\big([X, Y]\big) \,=\, \big[\underline{\rho}(X), \underline{\rho}(Y)\big] \quad , \tag{9.28}$$

for all $X, Y \in \mathfrak{su}(n)$. The Lie algebra representations associated with the $\mathsf{SU}(n)$ representations listed in Example 9.7 read concretely as follows:

1. *Trivial representation:* By definition, we have that

$$\rho_{\text{triv}}(e^X) = \text{id} \quad , \tag{9.29}$$

hence $\underline{\rho_{\text{triv}}}(X) = 0$ for all $X \in \mathfrak{su}(n)$. This Lie algebra representation is unitary.

2. *Defining representation:* Let us compute

$$\rho_{\text{def}}(e^X) = e^X = \text{id} + X + \mathcal{O}(X^2) \quad , \tag{9.30}$$

hence we have that $\underline{\rho_{\text{def}}}(X) = X$ for all $X \in \mathfrak{su}(n)$. This Lie algebra representation is unitary when $\mathbb{C}^n$ is endowed with its standard inner product $\langle v, v' \rangle := v^\dagger \, v'$. Indeed,

$$\langle \underline{\rho_{\text{def}}}(X)(v), v' \rangle + \langle v, \underline{\rho_{\text{def}}}(X)(v') \rangle = v^\dagger \, X^\dagger \, v' + v^\dagger \, X \, v' = 0 \quad , \tag{9.31}$$

for all $X \in \mathfrak{su}(n)$ and $v, v' \in \mathbb{C}^n$, where in the last step we used that $X^\dagger = -X$ is anti-Hermitian.

3. *Adjoint representation:* Let us compute

$$\rho_{\text{ad}}(e^X)(Y) := e^X \, Y \, e^{-X} = Y + X\,Y - Y\,X + \mathcal{O}(X^2) = Y + [X, Y] + \mathcal{O}(X^2) \quad , \tag{9.32}$$

hence we have that $\underline{\rho_{\text{ad}}}(X)(Y) = [X, Y]$ is given by the Lie bracket for all $X, Y \in \mathfrak{su}(n)$.

Note that the representations above also make sense when we replace $\mathsf{SU}(n)$ by the unitary Lie group $\mathsf{U}(n)$.

9.3 Non-Abelian Yang-Mills theory

After our brief tour through the world of Lie groups and Lie algebras, we are now equipped with the necessary technology to introduce one of the main players of the standard model of particle physics: *non-Abelian Yang-Mills theory*. This is a non-Abelian generalization, associated to a choice of Lie group G, of the electromagnetic potential from Chapter 6. To understand better what happens below, please make sure that you refresh the electromagnetic potential and its gauge symmetries from Section 6.1 before you continue reading the current section. To simplify our presentation, we shall consider only the cases of unitary groups $\mathsf{U}(n)$ and special unitary group $\mathsf{SU}(n)$ in the sense of Example 9.2. So that we do not get lost in confusing notations and case distinctions, we shall uniformly write G for any of these choices, i.e. in what follows the symbol G stands either for $\mathsf{U}(n)$ or $\mathsf{SU}(n)$ and the symbol $\mathfrak{g}$ stands for the corresponding Lie algebra from Example 9.4.

A *Yang-Mills field* is defined as a covector field

$$A : \mathbb{R}^d \longrightarrow \mathbb{R}^d \otimes \mathfrak{g} \ , \quad x \longmapsto A(x) \tag{9.33}$$

on the d-dimensional Minkowski spacetime $(\mathbb{R}^d, \eta)$ that takes values in the Lie algebra $\mathfrak{g}$ of the chosen Lie group G. Using our typical index notation A_μ for the components of covectors, this means that each component

$$A_\mu(x) = A^a_\mu(x)\, X_a \in \mathfrak{g} \tag{9.34}$$

takes values in the Lie algebra, hence it can be expanded as in Remark 9.5 by picking a basis $\{X_a \in \mathfrak{g}\}$. (From this we see that the number of components of a Yang-Mills field depends on the dimension of the underlying Lie algebra $\mathfrak{g}$.) For our examples given by $\mathfrak{g} = \mathfrak{u}(n)$ or $\mathfrak{g} = \mathfrak{su}(n)$ from Example 9.4, this means that $A_\mu(x)$ is an anti-Hermitian $n \times n$-matrix that is also trace-free in the case of $\mathfrak{g} = \mathfrak{su}(n)$.

The *gauge transformations* of Yang-Mills theory are labeled by functions

$$U : \mathbb{R}^d \longrightarrow G \ , \quad x \longmapsto U(x) \tag{9.35}$$

from the Minkowski spacetime to the Lie group G, and they act on Yang-Mills fields according to

$$T_U \ : \ A_\mu(x) \longmapsto (T_U A)_\mu(x) := U(x)\, A_\mu(x)\, U^{-1}(x) + U(x)\, \partial_\mu U^{-1}(x) \quad . \tag{9.36}$$

To simplify notation, we shall often suppress the arguments $x \in \mathbb{R}^d$ and simply write this as

$$(T_U A)_\mu \ = \ U\, A_\mu\, U^{-1} + U\, \partial_\mu U^{-1} \quad . \tag{9.37}$$

It is however important to keep in mind that all quantities in this expression depend on the spacetime point $x \in \mathbb{R}^d$, and in particular the gauge transformations are local (i.e. x-dependent) transformations. We recognize that the first term $U\, A_\mu\, U^{-1}$ is given by the adjoint representation of G on the Lie algebra $\mathfrak{g}$, see Example 9.7, but the second term is new and it relies on the fact that $U(x)$ is a function on spacetime. One can check that the transformation law (9.36) is a (non-linear, but affine) group representation, which means that $(T_{(U'U)}A)_\mu = \big(T_{U'}(T_U A)\big)_\mu$ for all gauge transformations U and U'. For completeness, let us do the relevant calculation

$$\begin{aligned}
\big(T_{U'}(T_U A)\big)_\mu &= U'\, (T_U A)_\mu\, U'^{-1} + U'\, \partial_\mu U'^{-1} = U' \Big(U\, A_\mu\, U^{-1} + U\, \partial_\mu U^{-1}\Big)\, U'^{-1} + U'\, \partial_\mu U'^{-1} \\
&= (U'\, U)\, A_\mu\, (U'\, U)^{-1} + (U'\, U)\, (\partial_\mu U^{-1})\, U'^{-1} + (U'\, U)\, U^{-1}\, \partial_\mu U'^{-1} \\
&= (U'\, U)\, A_\mu\, (U'\, U)^{-1} + (U'\, U)\, \partial_\mu (U'\, U)^{-1} = (T_{(U'U)}A)_\mu \quad ,
\end{aligned} \tag{9.38}$$

where in the third line we used the Leibniz rule $\partial_\mu (U'\, U)^{-1} = \partial_\mu (U^{-1}\, U'^{-1}) = (\partial_\mu U^{-1})\, U'^{-1} + U^{-1}\, \partial_\mu U'^{-1}$ for partial derivatives.

The *field strength tensor* of non-Abelian Yang-Mills theory is slightly more complicated than its analog in electromagnetism. The naive idea to consider simply the antisymmetrized partial derivative $\partial_\mu A_\nu - \partial_\nu A_\mu$ does not work for non-Abelian Lie groups G because it has an unpleasant transformation behavior under gauge transformations. This issue can be resolved by making use of the (in general non-trivial) Lie bracket on $\mathfrak{g}$ to define

$$F_{\mu\nu}(x) := \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + \big[A_\mu(x), A_\nu(x)\big] \quad . \tag{9.39}$$

By construction, the field strength tensor takes values in the Lie algebra $F_{\mu\nu}(x) \in \mathfrak{g}$ and it is antisymmetric in the two covector indices $F_{\mu\nu}(x) = -F_{\nu\mu}(x)$. Expanding the field strength tensor $F_{\mu\nu} = F^a_{\mu\nu}\, X_a$ in a basis $\{X_a \in \mathfrak{g}\}$ of the Lie algebra one finds

$$F_{\mu\nu} = \partial_\mu A^a_\nu\, X_a - \partial_\nu A^a_\mu\, X_a + A^b_\mu\, A^c_\nu\, [X_b, X_c] = \Big(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc}\, A^b_\mu\, A^c_\nu\Big)\, X_a \quad , \tag{9.40}$$

where f^a_{bc} are the structure constants from Remark 9.5. The main difference between the Yang-Mills and the electromagnetic field strength tensor is that the former is a non-linear expression in the A's, while the latter is linear. For the resulting QFT this implies that non-Abelian Yang-Mills fields have self-interactions, while photons do not. With a short calculation, which I leave for you as an exercise, one shows that $F_{\mu\nu}$ transforms under gauge transformations according to

$$T_U \;:\; F_{\mu\nu}(x) \longmapsto (T_U F)_{\mu\nu}(x) = U(x)\, F_{\mu\nu}(x)\, U^{-1}(x) \quad , \tag{9.41}$$

which we recognize as the adjoint representation from Example 9.7.

The construction of a *gauge invariant action functional* for Yang-Mills theory is now relatively simple. Even though the quantity $F^{\mu\nu}F_{\mu\nu}$ is valued in matrices (because $F_{\mu\nu}(x) \in \mathfrak{g}$ and $F^{\mu\nu} \in \mathfrak{g}$ are matrices), each component of these matrices transforms as a scalar under Poincaré transformations. We can produce from this a number by taking the trace $\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu})$, which again transforms as a scalar field under Poincaré transformations. Using cyclicity of the trace together with the transformation law (9.41), we compute

$$\begin{aligned} \mathrm{Tr}\Big((T_U F)^{\mu\nu}(T_U F)_{\mu\nu}\Big) &= \mathrm{Tr}\Big(U\, F^{\mu\nu}\, U^{-1}\, U\, F_{\mu\nu} U^{-1}\Big) \\ &= \mathrm{Tr}\Big(U^{-1}\, U\, F^{\mu\nu}\, F_{\mu\nu}\Big) = \mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) \quad , \end{aligned} \tag{9.42}$$

which means that $\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu})$ is gauge invariant. This allows us to define the gauge invariant and Poincaré invariant action functional

$$S_{\mathrm{YM}}[A] := \int_{\mathbb{R}^d} \frac{1}{2\, g^2_{\mathrm{YM}}}\, \mathrm{Tr}(F^{\mu\nu}\, F_{\mu\nu})\, \mathrm{d}x \quad , \tag{9.43}$$

where g_{YM} is a constant that is called the *Yang-Mills coupling constant*. The Yang-Mills action in the form of (9.43) is an efficient packaging of a rather rich action functional that includes terms that are quadratic, cubic and quartic in the field A_μ. In fact, using (9.39), we can write out the Yang-Mills action in terms of the A's and find

$$\begin{aligned} S_{\mathrm{YM}}[A] \;&=\; \int_{\mathbb{R}^d} \frac{1}{2\, g^2_{\mathrm{YM}}}\, \mathrm{Tr}\Big(\Big(\partial^\mu A^\nu - \partial^\nu A^\mu + [A^\mu, A^\nu]\Big)\, F_{\mu\nu}\Big)\, \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \frac{1}{2\, g^2_{\mathrm{YM}}}\, \mathrm{Tr}\Big(\Big(2\,\partial^\mu A^\nu + [A^\mu, A^\nu]\Big)\, F_{\mu\nu}\Big)\, \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \frac{1}{2\, g^2_{\mathrm{YM}}}\, \mathrm{Tr}\Big(2\,\partial^\mu A^\nu\, \partial_\mu A_\nu - 2\,\partial^\mu A^\nu\, \partial_\nu A_\mu + 4\,(\partial^\mu A^\nu)\,[A_\mu, A_\nu] + [A^\mu, A^\nu]\,[A_\mu, A_\nu]\Big)\, \mathrm{d}x \quad , \end{aligned} \tag{9.44}$$

where in the second line we used antisymmetry of $F_{\mu\nu} = -F_{\nu\mu}$ and in the last line we used cyclicity of the trace. Upon quantization, this will lead to a 3-valent and a 4-valent interaction vertex in the Feynman rules that describes the self-interactions of the Yang-Mills field.

Remark 9.12. To avoid confusion, I would like to add the following remarks:

(i) You might be worried by the fact that the coupling constant g_{YM} appears in (9.43) in front of the whole Lagrangian, including the free (i.e. quadratic) part. This is simply a convenient convention that we have used to streamline our formulas in this section. Using the field redefinition $\widetilde{A}_\mu := \frac{1}{g_{\text{YM}}} A_\mu$ for the Yang-Mills field, the action functional in the form of (9.44) can be rewritten as

$$S_{\text{YM}}[\widetilde{A}] = \int_{\mathbb{R}^d} \frac{1}{2} \operatorname{Tr}\Big(2\,\partial^\mu \widetilde{A}^\nu\, \partial_\mu \widetilde{A}_\nu - 2\,\partial^\mu \widetilde{A}^\nu\, \partial_\nu \widetilde{A}_\mu + 4\, g_{\text{YM}}\, (\partial^\mu \widetilde{A}^\nu)\, [\widetilde{A}_\mu, \widetilde{A}_\nu] + g_{\text{YM}}^2\, [\widetilde{A}^\mu, \widetilde{A}^\nu]\, [\widetilde{A}_\mu, \widetilde{A}_\nu] \Big)\, \mathrm{d}x \quad , \tag{9.45}$$

where the coupling constant is now at its usual spot in front of the interaction terms. This form of the action is the most convenient one for perturbation theory, because it allows for an obvious and simple counting of powers of the coupling constant.

(ii) You also might ask why the Yang-Mills action (9.43) has a numerical prefactor of $\frac{1}{2}$, while the related Maxwell action (6.11) has a numerical prefactor of $-\frac{1}{4}$. The relative minus sign is related to our anti-Hermiticity convention for the Lie algebras $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$ from Example 9.4, which implies that the trace $\operatorname{Tr}(X\,X) \leq 0$ is non-positive for any Lie algebra element X. The relative factor of 2 is introduced to match the usual convention that the basis $\{X_a \in \mathfrak{g}\}$ of the Lie algebra $\mathfrak{g} = \mathfrak{u}(n)$ or $\mathfrak{g} = \mathfrak{su}(n)$ is chosen such that it satisfies the orthonormality condition $\operatorname{Tr}(X_a\, X_b) = -\frac{1}{2}\,\delta_{ab}$. (You can check this explicitly for our rescaled Pauli matrices for $\mathfrak{su}(2)$ in Remark 9.5.) Hence, writing the field strength $F_{\mu\nu} = F^a_{\mu\nu}\, X_a$ in terms of such basis, we find that the action

$$S_{\text{YM}}[A] = \int_{\mathbb{R}^d} \frac{1}{2\, g_{\text{YM}}^2}\, F^a_{\mu\nu}\, F^{\mu\nu b}\, \operatorname{Tr}(X_a\, X_b)\, \mathrm{d}x = \int_{\mathbb{R}^d} -\frac{1}{4\, g_{\text{YM}}^2}\, F^a_{\mu\nu}\, F^{\mu\nu a}\, \mathrm{d}x \tag{9.46}$$

has a prefactor that is compatible with the one of the Maxwell action (6.11), up to the factor of $\frac{1}{g_{\text{YM}}^2}$ that we have explained in item (i) above.

Last but not least it is worthwhile to mention once more that different literature resources use different conventions. For instance, when working with the convention that the Lie algebra elements of $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$ are described by Hermitian matrices, then there will be factors of i in some of the formulas above. The conventions taken in these lecture notes try to minimize the appearance of unnecessary prefactors.

9.4 Coupling to matter fermions

The coupling of the Yang-Mills field from Section 9.3 to matter fields can be achieved by a gauging and minimal coupling procedure that is similar to the case of electromagnetism, see Section 7.1. The role that is played by the electric charge q in the context of QED is now played by a linear

representation $\rho : G \to \mathsf{GL}(V)$ (in the sense of Definition 9.6) of the Lie group G underlying the Yang-Mills theory. To simplify our presentation, we again assume that G is either the unitary group $\mathsf{U}(n)$ or the special unitary group $\mathsf{SU}(n)$ and we further assume that ρ is a unitary representation, e.g. the trivial or the defining representation from Example 9.7.

The first type of matter field we discuss is given by a Dirac field

$$\Psi \,:\, \mathbb{R}^d \longrightarrow \mathbb{C}^N \otimes V \;, \quad x \longmapsto \Psi(x) \tag{9.47}$$

on the d-dimensional Minkowski spacetime $(\mathbb{R}^d, \eta)$ that takes values in a unitary representation ρ of G on a complex vector space V. We demand that this field transforms under gauge transformations $U : \mathbb{R}^d \to G\,,\; x \mapsto U(x)$, according to the given representation, i.e.

$$T_U \,:\, \Psi(x) \longmapsto (T_U\Psi)(x) := \rho\big(U(x)\big)\,\Psi(x) \quad . \tag{9.48}$$

We shall again suppress most of the time the argument $x \in \mathbb{R}^d$ and simply write this as $T_U\Psi = \rho(U)\,\Psi$, but it is important to keep in mind that $U(x)$ is a local (i.e. x-dependent) transformation. From this transformation behavior we deduce that the Dirac adjoint field $\overline{\Psi} := \Psi^\dagger\,(\mathrm{i}\,\gamma^0)$ transforms according to

$$T_U\overline{\Psi} := \overline{T_U\Psi} = \big(\rho(U)\,\Psi\big)^\dagger\,(\mathrm{i}\,\gamma^0) = \Psi^\dagger\,\rho(U^{-1})\,(\mathrm{i}\,\gamma^0) = \Psi^\dagger\,(\mathrm{i}\,\gamma^0)\,\rho(U^{-1}) = \overline{\Psi}\,\rho(U^{-1}) \quad , \tag{9.49}$$

where in the third step we used that ρ is a unitary representation, hence $\rho(U)^\dagger = \big(\rho(U)\big)^{-1} = \rho(U^{-1})$. In the fourth step we used that $\rho(U^{-1})\,(\mathrm{i}\,\gamma^0) = (\mathrm{i}\,\gamma^0)\,\rho(U^{-1})$ commute because they act on different tensor factors of $\mathbb{C}^N \otimes V$. (When one is more pedantic, one could write $\mathrm{id} \otimes \rho(U^{-1})$ and $(\mathrm{i}\,\gamma^0) \otimes \mathrm{id}$ for these linear maps on $\mathbb{C}^N \otimes V$, but this would lead to a horrible notation.) Using the terminology of Remark 9.9, this means that the Dirac adjoint $\overline{\Psi}$ transforms in the dual representation of ρ. From this discussion it follows that the Dirac inner product is gauge invariant

$$\overline{\Psi}\,\chi \longmapsto \big(T_U\overline{\Psi}\big)\,(T_U\chi) = \overline{\Psi}\,\rho(U^{-1})\,\rho(U)\,\chi = \overline{\Psi}\,\chi \quad , \tag{9.50}$$

which is a useful observation that will later allow us to write down a mass term in the action.

Similarly to the case of electrodynamics, defining a gauge invariant kinetic term for Ψ is more problematic because, being a local transformation $U(x)$, the gauge transformations do not commute with partial derivatives, i.e. $\partial_\mu\big(\rho(U(x))\,\Psi(x)\big) \neq \rho(U(x))\,\partial_\mu\Psi(x)$. The solution of this problem lies again in introducing a *gauge covariant derivative*, which in the current non-Abelian case is given by

$$D_\mu\Psi(x) := \partial_\mu\Psi(x) + \underline{\rho}(A_\mu(x))\,\Psi(x) \quad , \tag{9.51}$$

where A_μ is the Yang-Mills field and $\underline{\rho}$ is the Lie algebra representation induced by ρ, see Example 9.11. One can show by a direct calculation that, under a combined gauge transformation on Ψ and A_μ according to (9.36) and (9.48), the gauge covariant derivative transforms as

$$T_U \,:\, D_\mu\Psi(x) \longmapsto \rho\big(U(x)\big)\,D_\mu\Psi(x) \quad . \tag{9.52}$$

For completeness, I will spell out this calculation, suppressing again the argument $x \in \mathbb{R}^d$ to improve readability

$$\begin{aligned} T_U(D_\mu\Psi) &= \partial_\mu(\rho(U)\,\Psi) + \underline{\rho}\Big(U\,A_\mu U^{-1} + U\,\partial_\mu U^{-1}\Big)\,\rho(U)\,\Psi \\ &= \rho(U)\,\partial_\mu\Psi + (\partial_\mu\rho(U))\,\Psi + \rho(U)\,\underline{\rho}(A_\mu)\,\Psi + \rho(U)\,(\partial_\mu\rho(U^{-1}))\,\rho(U)\,\Psi \\ &= \rho(U)\,\partial_\mu\Psi + \rho(U)\,\underline{\rho}(A_\mu)\,\Psi = \rho(U)\,D_\mu\Psi \quad . \end{aligned} \tag{9.53}$$

In the second step we used that $\underline{\rho}(U\,A_\mu\,U^{-1}) = \rho(U)\,\underline{\rho}(A_\mu)\,\rho(U^{-1})$ and $\underline{\rho}(U\,\partial_\mu U^{-1}) = \rho(U)\,\partial_\mu\rho(U^{-1})$, which follows from our construction of the Lie algebra representation $\underline{\rho}$ in Example 9.11. The last step uses the Leibniz rule in order to write $(\partial_\mu\rho(U^{-1}))\,\rho(U) = \partial_\mu(\rho(U^{-1})\,\rho(U)) - \rho(U^{-1})\,\partial_\mu\rho(U) = \partial_\mu\mathrm{id} - \rho(U^{-1})\,\partial_\mu\rho(U) = -\rho(U^{-1})\,\partial_\mu\rho(U)$.

These are now all the ingredients we need to write down the action functional

$$S_{\mathrm{Dirac}}[\Psi,\overline{\Psi},A] := \int_{\mathbb{R}^d} -\overline{\Psi}(\not{D}+m)\Psi\,\mathrm{d}x = \int_{\mathbb{R}^d} -\overline{\Psi}(\not{\partial}+\underline{\rho}(\not{A})+m)\Psi\,\mathrm{d}x \tag{9.54}$$

for a massive Dirac field taking values in a unitary representation ρ of G. This action is gauge invariant under the combined gauge transformations

$$T_U\Psi = \rho(U)\,\Psi \quad , \quad T_U\overline{\Psi} = \overline{\Psi}\,\rho(U^{-1}) \quad , \quad (T_U A)_\mu = U\,A_\mu U^{-1} + U\,\partial_\mu U^{-1} \tag{9.55}$$

on all the fields. We observe that there is a cubic interaction term, which upon quantization will lead to a 3-valent interaction vertex in the Feynman rules that is similar to the photon-Dirac-anti-Dirac vertex of QED from Section 7.2. The physical details of this interaction will of course depend on the choice of representation ρ; in particular, for the trivial representation we would have that $\underline{\rho_{\mathrm{triv}}} = 0$ is zero, hence this term disappears in this very special case. (This is analogous to the fact that the photon does couple trivially to particles of charge $q = 0$.) To obtain a full theory that describes both the Dirac and the Yang-Mills field, we simply add the two actions (9.43) and (9.54) and define

$$\begin{aligned} S_{\mathrm{YM+Dirac}}[\Psi,\overline{\Psi},A] &:= S_{\mathrm{Dirac}}[\Psi,\overline{\Psi},A] + S_{\mathrm{YM}}[A] \\ &= \int_{\mathbb{R}^d}\Big(-\overline{\Psi}(\not{D}+m)\Psi + \frac{1}{2\,g_{\mathrm{YM}}^2}\,\mathrm{Tr}(F^{\mu\nu}\,F_{\mu\nu})\Big)\,\mathrm{d}x \quad . \end{aligned} \tag{9.56}$$

This action is of course gauge invariant under the combined gauge transformations (9.55) because the two individual summands are gauge invariant.

Chiral fermions in $d = 4$ spacetime dimensions: In what follows we consider the d=4-dimensional Minkowski spacetime $(\mathbb{R}^4,\eta)$. An important physical feature of the standard model of particle physics (which is defined in $d = 4$ spacetime dimensions) is that the left and right-handed chiral components of Dirac spinors participate differently in the interactions mediated by the gauge bosons. From the representation theoretic perspective established in this section, this means that the left-handed Weyl fields Ψ_L live in a different representation of the Lie group than

the right-handed Weyl fields Ψ_R. (See around Eqn. (5.33) for the definition of left/right-handed Weyl spinors.) We take this into account by writing

$$\Psi_{L/R} : \mathbb{R}^4 \longrightarrow \mathbb{C}^2 \otimes V_{L/R} \ , \quad x \longmapsto \Psi_{L/R}(x) \tag{9.57}$$

for a left/right-handed Weyl field taking values in a unitary representation $\rho_{L/R}$ of the Lie group G on a complex vector space $V_{L/R}$. (In general, these representations will be different!) The corresponding gauge transformations from (9.48) then read as

$$(T_U \Psi_{L/R})(x) \ = \ \rho_{L/R}(U(x))\, \Psi_{L/R}(x) \tag{9.58}$$

and the gauge covariant derivatives from (9.51) are given by

$$D_\mu \Psi_{L/R}(x) \ = \ \partial_\mu \Psi_{L/R}(x) + \underline{\rho_{L/R}}(A_\mu(x))\, \Psi_{L/R}(x) \quad . \tag{9.59}$$

This all looks pretty much the same as for Dirac fields, so what's the problem? The problem arises when we want to write down an action functional similar to (9.54) for the chiral fields. To see this, let us recall the 4-dimensional gamma-matrices from (5.25), which we shall write compactly in the following block matrix form

$$\gamma^\mu = -\mathrm{i} \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \quad , \tag{9.60}$$

where $\sigma^0 = \overline{\sigma}^0 = 1_{2\times 2}$ is the 2×2 identity matrix and $\sigma^i = -\overline{\sigma}^i$ are the Pauli matrices. The Dirac inner product then reads in terms of the chiral components as

$$\overline{\Psi}\, \chi = \Psi^\dagger\, (\mathrm{i}\, \gamma^0)\, \chi = \begin{pmatrix} \Psi_L^\dagger & \Psi_R^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \Psi_L^\dagger\, \chi_R + \Psi_R^\dagger\, \chi_L \quad , \tag{9.61}$$

i.e. it mixes between the left and the right-handed chiral components. This means that this expression is in general *not* gauge invariant whenever the two representations $\rho_{L/R}$ are different, which is however the characteristic feature of a chiral gauge theory. Explicitly, under a gauge transformation we have in general that

$$\overline{\Psi}\, \chi \ \longmapsto \ \Psi_L^\dagger\, \rho_L(U^{-1})\, \rho_R(U)\, \chi_R + \Psi_R^\dagger\, \rho_R(U^{-1})\, \rho_L(U)\, \chi_L \ \neq \ \overline{\Psi}\, \chi \quad . \tag{9.62}$$

So mass terms are problematic in chiral gauge theories, but what about kinetic terms? These are much better, because

$$\begin{aligned} \overline{\Psi}\, \gamma^\mu\, \chi = \Psi^\dagger\, (\mathrm{i}\, \gamma^0)\, \gamma^\mu\, \chi = -\mathrm{i} \begin{pmatrix} \Psi_L^\dagger & \Psi_R^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} \\ = -\mathrm{i} \begin{pmatrix} \Psi_L^\dagger & \Psi_R^\dagger \end{pmatrix} \begin{pmatrix} \overline{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = -\mathrm{i} \left(\Psi_L^\dagger\, \overline{\sigma}^\mu\, \chi_L + \Psi_R^\dagger\, \sigma^\mu\, \chi_R \right) \end{aligned} \tag{9.63}$$

does not mix between the left and the right-handed chiral components, hence it is gauge invariant. This allows us to define the gauge invariant action functional

$$S_{\text{chiral}}[\Psi,\overline{\Psi},A] \;:=\; \int_{\mathbb{R}^4} \mathrm{i}\Big(\Psi_L^\dagger\,\overline{\sigma}^\mu D_\mu\,\Psi_L + \Psi_R^\dagger\,\sigma^\mu D_\mu\,\Psi_R\Big)\,\mathrm{d}x \tag{9.64}$$

that consists only of a kinetic term with gauge covariant derivative. The cubic interaction term in this action will lead again, upon quantization, to a 3-valent interaction vertex in the Feynman rules. We can further add this to the Yang-Mills action (9.43) and obtain the action functional

$$\begin{aligned} S_{\text{YM+chiral}}[\Psi,\overline{\Psi},A] \;&:=\; S_{\text{chiral}}[\Psi,\overline{\Psi},A] + S_{\text{YM}}[A] \\ &= \int_{\mathbb{R}^4} \Big(\mathrm{i}\,\Psi_L^\dagger\,\overline{\sigma}^\mu D_\mu\,\Psi_L + \mathrm{i}\,\Psi_R^\dagger\,\sigma^\mu D_\mu\,\Psi_R + \frac{1}{2\,g_{\text{YM}}^2}\,\mathrm{Tr}\big(F^{\mu\nu}\,F_{\mu\nu}\big)\Big)\,\mathrm{d}x \end{aligned} \tag{9.65}$$

that is gauge invariant under the combined gauge transformations

$$T_U\Psi_{L/R} = \rho_{L/R}(U)\,\Psi_{L/R} \quad , \tag{9.66a}$$

$$T_U\Psi_{L/R}^\dagger = \Psi_{L/R}^\dagger\,\rho_{L/R}(U^{-1}) \quad , \tag{9.66b}$$

$$(T_U A)_\mu = U\,A_\mu U^{-1} + U\,\partial_\mu U^{-1} \quad . \tag{9.66c}$$

It is important to emphasize again that a standard mass term $m\,\overline{\Psi}\Psi$ is forbidden in this action because it is in general not gauge invariant for $\rho_L \neq \rho_R$. The standard model of particle physics comes with a clever way to nevertheless introduce a suitable mass term, which is through the Higgs mechanism for spontaneous gauge symmetry breaking.

9.5 Higgs sector

Throughout this section we assume again that G is either the unitary group $\mathsf{U}(n)$ or the special unitary group $\mathsf{SU}(n)$. A *Higgs field* is a complex scalar field

$$\Phi \,:\, \mathbb{R}^d \longrightarrow \mathbb{C}\otimes V \;,\quad x \longmapsto \Phi(x) \tag{9.67}$$

on the d-dimensional Minkowski spacetime $(\mathbb{R}^d,\eta)$ that takes values in a unitary representation ρ of G on a complex vector space V. We demand that this field transforms under gauge transformations $U:\mathbb{R}^d\to G\,,\; x\mapsto U(x)$ according to the given representation, i.e.

$$T_U \,:\, \Phi(x) \longmapsto (T_U\Phi)(x) \;:=\; \rho\big(U(x)\big)\,\Phi(x) \quad , \tag{9.68}$$

which by unitarity of ρ implies that the adjoint Higgs field $\Phi^\dagger$ transforms as

$$T_U \,:\, \Phi^\dagger(x) \longmapsto (T_U\Phi^\dagger)(x) \;=\; \Phi^\dagger(x)\,\rho(U^{-1}(x)) \quad . \tag{9.69}$$

We shall again suppress most of the time the argument $x \in \mathbb{R}^d$ and simply write this as $T_U\Phi = \rho(U)\,\Phi$ and $T_U\Phi^\dagger = \Phi^\dagger\,\rho(U^{-1})$.

Our definition (9.51) of the gauge covariant derivative for Dirac fields applies ad verbum to the Higgs field Φ and its adjoint $\Phi^\dagger$, which yields

$$D_\mu\Phi(x) := \partial_\mu\Phi(x) + \underline{\rho}(A_\mu(x))\,\Phi(x) \quad , \tag{9.70a}$$
$$D_\mu\Phi^\dagger(x) := \partial_\mu\Phi^\dagger(x) - \Phi^\dagger(x)\,\underline{\rho}(A_\mu(x)) \quad . \tag{9.70b}$$

Note that these formulas are related by taking adjoints, i.e. $(D_\mu\Phi)^\dagger = D_\mu\Phi^\dagger$, because ρ is by hypothesis a unitary representation. By the same type of calculation as in (9.53), one checks that under a combined gauge transformation

$$T_U\Phi = \rho(U)\,\Phi \quad , \quad T_U\Phi^\dagger = \Phi^\dagger\,\rho(U^{-1}) \quad , \quad (T_U A)_\mu = U\,A_\mu\,U^{-1} + U\,\partial_\mu U^{-1} \tag{9.71}$$

on Φ, $\Phi^\dagger$ and A_μ these covariant derivatives transform according to

$$T_U(D_\mu\Phi) = \rho(U)\,D_\mu\Phi \quad , \quad T_U(D_\mu\Phi^\dagger) = D_\mu\Phi^\dagger\,\rho(U^{-1}) \quad . \tag{9.72}$$

From this observation it follows that a kinetic term of the form $D^\mu\Phi^\dagger\,D_\mu\Phi$ is gauge invariant, and so are potential terms of the form $(\Phi^\dagger\,\Phi)^n$.

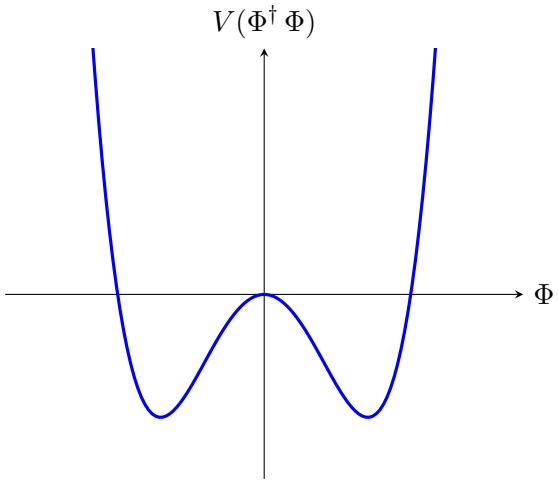
With these building blocks we can construct a gauge invariant action functional for the Higgs field. The typical choice for the *Higgs action functional* is given by

$$S_{\text{Higgs}}[\Phi,\Phi^\dagger,A] := \int_{\mathbb{R}^d} -\Big(D^\mu\Phi^\dagger\,D_\mu\Phi - \mu^2\,\Phi^\dagger\,\Phi + \lambda\,(\Phi^\dagger\,\Phi)^2\Big)\,\mathrm{d}x \quad , \tag{9.73}$$

where $\mu^2 > 0$ and $\lambda > 0$ are positive real parameters. Comparing this with the usual complex Klein-Gordon action in (2.17), we note that the mass term comes with a "wrong sign". It is important to stress that this is not a typo or an accident, but it is the most essential feature of the Higgs field! Let me explain what happens as a consequence of this "wrong sign" mass term: Note that the Higgs potential

$$V(\Phi^\dagger\,\Phi) = -\mu^2\,\Phi^\dagger\,\Phi + \lambda\,(\Phi^\dagger\,\Phi)^2 \tag{9.74a}$$

has the following shape



(9.74b)

from which we see that it has multiple minima. This means that the Higgs field has multiple classical vacua, which are, by definition, constant field configurations $\Phi(x) \equiv \Phi_0 \in \mathbb{C} \otimes V$ that minimize the potential. Note that each of these vacua is in general *not* invariant under gauge transformations, i.e. $\rho(U)\,\Phi_0 \neq \Phi_0$, which means that they break gauge symmetry. This phenomenon is called *spontaneous gauge symmetry breaking*, where the term spontaneous refers to the fact that, even though the action (9.73) is gauge invariant, the individual vacuum solutions of the resulting equations of motion are not. This mechanism of symmetry breaking is used in the standard model of particle physics to generate mass terms for matter fermions and for the W^+, W^- and Z gauge bosons. This will be discussed in more detail in some of the student projects/presentations.

We finish this section with the observation that we can add to (9.73) the Yang-Mills action (9.43) and thereby obtain the action functional

$$\begin{aligned} S_{\text{YM+Higgs}}[\Phi, \Phi^\dagger, A] \;&:=\; S_{\text{Higgs}}[\Phi, \Phi^\dagger, A] + S_{\text{YM}}[A] \\ &= \int_{\mathbb{R}^d} \Big(- D^\mu \Phi^\dagger\, D_\mu \Phi + \mu^2\, \Phi^\dagger\, \Phi - \lambda\, (\Phi^\dagger\, \Phi)^2 + \frac{1}{2\, g_{\text{YM}}^2} \operatorname{Tr}(F^{\mu\nu}\, F_{\mu\nu}) \Big)\, \mathrm{d}x \end{aligned} \tag{9.75}$$

for the so-called Yang-Mills-Higgs field theory. This combined action functional is invariant under the combined gauge transformations (9.71) acting on all fields.

9.6 Action functional for the standard model

We have now all the necessary techniques to write down the action functional for the standard model of particle physics. Throughout the whole section we work on the $d = 4$ dimensional Minkowski spacetime $(\mathbb{R}^4, \eta)$ and fix $G = \mathsf{U}(1) \times \mathsf{SU}(2) \times \mathsf{SU}(3)$ to be the Lie group of the standard model. Note that the elements of this product Lie group are triples $(U_1, U_2, U_3) \in \mathsf{U}(1) \times \mathsf{SU}(2) \times \mathsf{SU}(3)$ consisting of elements of the individual factors, i.e. $U_1 \in \mathsf{U}(1)$ is a unitary 1×1-matrix, $U_2 \in \mathsf{SU}(2)$ is a special unitary 2×2-matrix and $U_3 \in \mathsf{SU}(3)$ is a special unitary 3×3-matrix.

The action functional for the standard model takes the following form

$$S_{\text{SM}} \;=\; S_{\text{gauge}} + S_{\text{matter}} + S_{\text{Higgs}} + S_{\text{Yukawa}} \quad , \tag{9.76}$$

where each of the individual terms can be constructed using the techniques developed in this chapter. Since the final result for S_{SM} is quite lengthy, due to the many different particles/fields of the standard model (see the graphical visualization in Section 9.1), it makes sense to present the individual terms step by step. This is precisely what we shall do in the paragraphs below.

The action S_{gauge}: Since the standard model Lie group $G = \mathsf{U}(1) \times \mathsf{SU}(2) \times \mathsf{SU}(3)$ is a product group, its Lie algebra decomposes into a direct sum $\mathfrak{g} = \mathfrak{u}(1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(3)$. This means that a Yang-Mills field A_μ for the Lie group G can be decomposed into three different components, which we shall denote as follows:

(1) The $\mathsf{U}(1)$ Yang-Mills field will be denoted by $B_\mu \in \mathfrak{u}(1)$ and its field strength by $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] = \partial_\mu B_\nu - \partial_\nu B_\mu$, where the last identity holds because $\mathfrak{u}(1)$ is an Abelian Lie algebra, i.e. its Lie bracket is trivial.

(2) The $\mathsf{SU}(2)$ Yang-Mills field will be denoted by $W_\mu \in \mathfrak{su}(2)$ and its field strength by $W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + [W_\mu, W_\nu]$.

(3) The $\mathsf{SU}(3)$ Yang-Mills field will be denoted by $G_\mu \in \mathfrak{su}(3)$ and its field strength by $G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu + [G_\mu, G_\nu]$.

Under a gauge transformation $U = (U_1, U_2, U_3) : \mathbb{R}^d \to \mathsf{U}(1) \times \mathsf{SU}(2) \times \mathsf{SU}(3)$ these components transform according to (9.36) as

$$(T_U B)_\mu = U_1\, B_\mu U_1^{-1} + U_1\, \partial_\mu U_1^{-1} = B_\mu + U_1\, \partial_\mu U_1^{-1} \quad , \tag{9.77a}$$

$$(T_U W)_\mu = U_2\, W_\mu U_2^{-1} + U_2\, \partial_\mu U_2^{-1} \quad , \tag{9.77b}$$

$$(T_U G)_\mu = U_3\, G_\mu U_3^{-1} + U_3\, \partial_\mu U_3^{-1} \quad , \tag{9.77c}$$

i.e. the each component transforms according to its corresponding factor of the product group. The action S_{gauge} for the gauge bosons of the standard model is then given by summing the Yang-Mills actions (9.43) for the three components, i.e.

$$S_{\text{gauge}} = \int_{\mathbb{R}^4} \left(\frac{1}{2\, g_1^2} \operatorname{Tr}(B^{\mu\nu}\, B_{\mu\nu}) + \frac{1}{2\, g_2^2} \operatorname{Tr}(W^{\mu\nu}\, W_{\mu\nu}) + \frac{1}{2\, g_3^2} \operatorname{Tr}(G^{\mu\nu}\, G_{\mu\nu}) \right) \mathrm{d}x \quad , \tag{9.78}$$

where $g_1, g_2, g_3 \in \mathbb{R}$ are three independent coupling constants for the three factors of $G = \mathsf{U}(1) \times \mathsf{SU}(2) \times \mathsf{SU}(3)$ whose values have to be fixed by experiment. This action is invariant under the gauge transformations (9.77).

The action S_{matter}: This part of the action describes the matter fermions of the standard model. As we have seen in Section 9.4, the coupling of matter fermions to Yang-Mills fields can be described by choosing a unitary representation ρ of the Lie group G on some complex vector space V. Since

$G = \mathsf{U}(1) \times \mathsf{SU}(2) \times \mathsf{SU}(3)$ is a product Lie group, we can build such representations from a unitary representation ρ_1 of $\mathsf{U}(1)$, a unitary representation ρ_2 of $\mathsf{SU}(2)$ and a unitary representation ρ_3 of $\mathsf{SU}(3)$. Explicitly, denoting the underlying vector spaces of these representations by V_1, V_2 and V_3, one checks that

$$\begin{aligned} \rho \,:\, \mathsf{U}(1) \times \mathsf{SU}(2) \times \mathsf{SU}(3) &\longrightarrow \mathsf{GL}(V_1 \otimes V_2 \otimes V_3) \quad , \\ (U_1, U_2, U_3) &\longmapsto \rho(U_1, U_2, U_3) := \rho_1(U_1) \otimes \rho_2(U_2) \otimes \rho_3(U_3) \end{aligned} \tag{9.79}$$

defines a unitary representation. A left or right-handed Weyl field $\Psi_{L/R} : \mathbb{R}^4 \to \mathbb{C}^2 \otimes V_1 \otimes V_2 \otimes V_3$ that takes values in such representation then transforms under gauge transformations as

$$T_U \Psi_{L/R} \;=\; \rho_1(U_1)\, \rho_2(U_2)\, \rho_3(U_3)\, \Psi_{L/R} \quad , \tag{9.80}$$

where on the right-hand side we suppressed the tensor products in $\rho_1(U_1) \otimes \rho_2(U_2) \otimes \rho_3(U_3)$ to improve readability. The corresponding gauge covariant derivative reads as

$$D_\mu \Psi_{L/R} \;=\; \partial_\mu \Psi_{L/R} + \underline{\rho_1}(B_\mu)\, \Psi_{L/R} + \underline{\rho_2}(W_\mu)\, \Psi_{L/R} + \underline{\rho_3}(G_\mu)\, \Psi_{L/R} \quad , \tag{9.81}$$

where B_μ, W_μ and G_μ denote the three components of the Yang-Mills field that correspond to the three factors of $G = \mathsf{U}(1) \times \mathsf{SU}(2) \times \mathsf{SU}(3)$.

So what we have to do is specify the various representations for the different standard model matter fermions in the graphical table from Section 9.1. It is important to emphasize that these representations can not be predicted theoretically, but they had to be determined by experiment. Recalling the specific representations from Examples 9.7 and 9.8, we start by listing the relevant representations for the matter fermions from the first generation:

	$\mathsf{U}(1)$ rep. ρ_1	$\mathsf{SU}(2)$ rep. ρ_2	$\mathsf{SU}(3)$ rep. ρ_3	Abbreviation
Left-handed quark q_L	$\rho^{(1)}$	ρ_{def}	ρ_{def}	$(\frac{1}{3}, \mathbf{2}, \mathbf{3})$
Right-handed up quark u_R	$\rho^{(4)}$	ρ_{triv}	ρ_{def}	$(\frac{4}{3}, \mathbf{1}, \mathbf{3})$
Right-handed down quark d_R	$\rho^{(-2)}$	ρ_{triv}	ρ_{def}	$(-\frac{2}{3}, \mathbf{1}, \mathbf{3})$
Left-handed lepton ℓ_L	$\rho^{(-3)}$	ρ_{def}	ρ_{triv}	$(-1, \mathbf{2}, \mathbf{1})$
Right-handed electron e_R	$\rho^{(-6)}$	ρ_{triv}	ρ_{triv}	$(-2, \mathbf{1}, \mathbf{1})$

The last column is a useful and efficient way to label the relevant representations that you will often find in the literature. The first entry of the triple is called *weak hypercharge* and it is computed from the $\mathsf{U}(1)$ representation $\rho^{(k)}$ by $\frac{k}{3}$. The second entry describes the $\mathsf{SU}(2)$ representation and it is $\mathbf{1}$ for the 1-dimensional trivial representation ρ_{triv} and $\mathbf{2}$ for the 2-dimensional defining representation ρ_{def}. So the bold number indicates the dimension of the representation. The third entry describes the $\mathsf{SU}(3)$ representation and it is $\mathbf{1}$ for the 1-dimensional trivial representation ρ_{triv} and $\mathbf{3}$ for the 3-dimensional defining representation ρ_{def}.

It is important to note that the left-handed quark q_L lives in the 2-dimensional $\mathsf{SU}(2)$ defining representation ρ_{def}, i.e. it consists of two components that can be identified respectively with the left-handed up quark u_L and the left-handed down quark d_L. Similarly, the left-handed lepton

ℓ_L lives in the 2-dimensional $\mathsf{SU}(2)$ defining representation and its two components define the left-handed electron e_L and the left-handed neutrino ν_L. In our version of the standard model of particle physics, which agrees with the one that is typically presented in textbooks, there is no right-handed neutrino ν_R. Evidence from more recent experiments (e.g. neutrino oscillations) suggests that right-handed neutrinos exist, hence one should think about adding ν_R to the table above. If it exist, one knows that the relevant representation for ν_R must be $(0, \mathbf{1}, \mathbf{1})$, which means that the right-handed neutrino participates trivially in all of the three fundamental interactions. (That's why right-handed neutrinos are sometimes called *sterile neutrinos*.) Right-handed neutrinos will be discussed further in some of the student projects/presentations.

The representations for the second and the third generation of matter fermions are identical to the ones for the first generation because, besides different masses, the particles from different generations have identical properties. This means that we have to introduce three versions of the fields listed in the table above, which we shall do by introducing a subscript $_i$ that runs over the three generations $i = 1, 2, 3$. The action S_{matter} for the matter fermions is then given by a sum over the chiral actions (9.64) that takes into account all particles and all generations. Explicitly, this reads as

$$S_{\text{matter}} = \sum_{i=1}^{3} \int_{\mathbb{R}^4} \Big(\mathrm{i}\, q_{L\,i}^\dagger\, \overline{\sigma}^\mu D_\mu\, q_{L\,i} + \mathrm{i}\, u_{R\,i}^\dagger\, \sigma^\mu D_\mu\, u_{R\,i} + \mathrm{i}\, d_{R\,i}^\dagger\, \sigma^\mu D_\mu\, d_{R\,i} + \mathrm{i}\, \ell_{L\,i}^\dagger\, \overline{\sigma}^\mu D_\mu\, \ell_{L\,i} + \mathrm{i}\, e_{R\,i}^\dagger\, \sigma^\mu D_\mu\, e_{R\,i} \Big)\, \mathrm{d}x \quad . \tag{9.82}$$

This action is invariant under the combined gauge transformations given by (9.77) and (9.80). Physically, it describes the dynamics of the matter fermions and their interaction with the gauge bosons.

The action S_{Higgs}: To introduce a Higgs field, one has to choose another unitary representation of the standard model Lie group $G = \mathsf{U}(1) \times \mathsf{SU}(2) \times \times \mathsf{SU}(3)$, which also has to be determined from experiment. Given any representation of the form $\rho = \rho_1 \otimes \rho_2 \otimes \rho_3$, the gauge transformation law for the Higgs field reads as

$$T_U \Phi \;=\; \rho_1(U_1)\, \rho_2(U_2)\, \rho_3(U_3)\, \Phi \quad , \quad T_U \Phi^\dagger \;=\; \Phi^\dagger\, \rho_1(U_1^{-1})\, \rho_2(U_2^{-1})\, \rho_3(U_3^{-1}) \tag{9.83}$$

and the corresponding gauge covariant derivatives are given by

$$D_\mu \Phi \;=\; \partial_\mu \Phi + \underline{\rho_1}(B_\mu)\, \Phi + \underline{\rho_2}(W_\mu)\, \Phi + \underline{\rho_3}(G_\mu)\, \Phi \quad , \tag{9.84a}$$

$$D_\mu \Phi^\dagger \;=\; \partial_\mu \Phi^\dagger - \Phi^\dagger\, \underline{\rho_1}(B_\mu) - \Phi^\dagger\, \underline{\rho_2}(W_\mu) - \Phi^\dagger\, \underline{\rho_3}(G_\mu) \quad . \tag{9.84b}$$

The standard model Higgs field comes in the following representation:

	$\mathsf{U}(1)$ rep. ρ_1	$\mathsf{SU}(2)$ rep. ρ_2	$\mathsf{SU}(3)$ rep. ρ_3	Abbreviation
Higgs field Φ	$\rho^{(-3)}$	ρ_{def}	ρ_{triv}	$(-1, \mathbf{2}, \mathbf{1})$

The action S_{Higgs} for the Higgs field is then given by (9.73), which for completeness will be repeated here

$$S_{\text{Higgs}} = \int_{\mathbb{R}^4} -\Big(D^\mu \Phi^\dagger \, D_\mu \Phi - \mu^2 \, \Phi^\dagger \, \Phi + \lambda \, (\Phi^\dagger \, \Phi)^2 \Big) \, \mathrm{d}x \quad . \tag{9.85}$$

This action is invariant under the combined gauge transformations given by (9.77) and (9.83). Physically, it describes the dynamics of the Higgs boson and its interaction with the gauge bosons. After spontaneous symmetry breaking, it will generate mass terms for the W^+, W^- and Z bosons.

The action S_{Yukawa}: The last term in the standard model action (9.76) is an interaction term (called *Yukawa interaction*) between the Higgs field and the matter fermions. From a physical perspective this is needed to introduce fermion masses through spontaneous gauge symmetry breaking. The existence of the Yukawa interaction terms relies very sensitively on the specific details of the representations given to the matter fermions and the Higgs field (see the tables above). The best way I know to write down the Yukawa terms is to use the following partial index notation for only the $\mathsf{SU}(2)$ representations: Since the left-handed fermions, i.e. q_L and ℓ_L, and the Higgs Φ live in the 2-dimensional $\mathsf{SU}(2)$ defining representation ρ_{def}, we can pick a basis of $\mathbb{C}^2$ and write $q_{L\,a}$, $\ell_{L\,a}$ and Φ_a, with $a = 1, 2$, for the two components. The other matter fermions u_R, d_R and e_R live in the 1-dimensional trivial $\mathsf{SU}(2)$ representation, hence they do not get such an index related to $\mathsf{SU}(2)$. I claim that the term

$$q_L^\dagger \, u_R \, \Phi := q_{L\,a}^\dagger \, u_R \, \Phi_a \qquad \text{(summation over } a = 1, 2 \text{ understood)} \tag{9.86}$$

is invariant under all $G = \mathsf{U}(1) \times \mathsf{SU}(2) \times \mathsf{SU}(3)$ gauge transformations. This can be shown by considering the individual factors separately. Starting with $U_3 \in SU(3)$, we use that Φ transforms in the trivial representation while q_L and u_R transform in the defining representation, hence

$$T_{U_3} \, : \, q_L^\dagger \, u_R \, \Phi \longmapsto \big(\rho_{\text{def}}(U_3) q_L\big)^\dagger \rho_{\text{def}}(U_3) u_R \, \Phi = q_L^\dagger \, u_R \, \Phi \quad , \tag{9.87}$$

where in the last step we used that the defining representation is unitary. Considering now $U_1 \in \mathsf{U}(1)$, we look up the corresponding $\mathsf{U}(1)$ representations from the table above and compute

$$T_{U_1} \, : \, q_L^\dagger \, u_R \, \Phi \longmapsto \rho^{(-1)}(U_1) \, \rho^{(4)}(U_1) \, \rho^{(-3)}(U_1) \, q_L^\dagger \, u_R \, \Phi \, = \, U_1^{-1+4-3} \, q_L^\dagger \, u_R \, \Phi \, = \, q_L^\dagger \, u_R \, \Phi \quad , \tag{9.88}$$

where we used that $\big(\rho^{(1)}(U_1) \, q_L\big)^\dagger = \big(U_1 \, q_L\big)^\dagger = U_1^{-1} \, q_L^\dagger$ because $U_1^\dagger = U_1^{-1}$ is a unitary 1×1-matrix. Finally, for $U_2 \in \mathsf{SU}(2)$ we look again at the corresponding representations from the table above and compute in index notation

$$T_{U_2} \, : \, q_{L\,a}^\dagger \, u_R \, \Phi_a \longmapsto \big(U_{2ab} \, q_{L\,b}\big)^\dagger \, u_R \, U_{2ac} \, \Phi_c = U_{2ab}^* \, U_{2ac} \, q_{L\,b}^\dagger \, u_R \, \Phi_c = q_{L\,b}^\dagger \, u_R \, \Phi_b \quad , \tag{9.89}$$

where in the last step we used that U_2 is a unitary matrix, i.e. $U_2^\dagger \, U_2 = \mathbf{1}$, which reads in index notation as $U_{2ab}^* \, U_{2ac} = \delta_{bc}$. All of this together shows that the term (9.86) is gauge invariant under $G = \mathsf{U}(1) \times \mathsf{SU}(2) \times \mathsf{SU}(3)$.

Let us now consider a second term of a similar form

$$q_L^\dagger \, d_R \, \widetilde{\Phi} := \epsilon_{ab} \, q_{L\,a}^\dagger \, d_R \, \Phi_b^* \qquad \text{(summation over } a, b = 1, 2 \text{ understood)} \quad , \tag{9.90}$$

where Φ_b^* denotes the complex conjugate of the b-component of the Higgs field and ϵ_{ab} are the entries of the following 2×2-matrix (the 2-dimensional epsilon-tensor)

$$\epsilon \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad . \tag{9.91}$$

Invariance of this term under $\mathsf{U}(1)$ and $\mathsf{SU}(3)$ gauge transformations is easy to check by looking again at the table above. Checking invariance under $\mathsf{SU}(2)$ gauge transformations is slightly more complicated. Given $U_2 \in \mathsf{SU}(2)$, we compute in index notation

$$T_{U_2} \;:\; \epsilon_{ab}\, q_{L\,a}^\dagger\, d_R\, \Phi_b^* \;\longmapsto\; \epsilon_{ab}\, U_{2\,ac}^*\, U_{2\,bd}^*\, q_{L\,c}^\dagger\, d_R\, \Phi_d^* \quad . \tag{9.92}$$

This means that the term (9.90) is gauge invariant under $\mathsf{SU}(2)$ if and only if the matrix identity

$$U_2^\dagger\, \epsilon\, U_2^* \;=\; \epsilon \tag{9.93}$$

holds true, where U_2^* denotes entry-wise complex conjugation (without transposition!). One can show that every special unitary 2×2-matrix can be written as

$$U_2 = \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix} \tag{9.94}$$

with some complex numbers $z, w \in \mathbb{C}$ satisfying $|z|^2 + |w|^2 = 1$. From this we check

$$U_2^\dagger\, \epsilon\, U_2^* \;=\; \begin{pmatrix} z^* & -w \\ w^* & z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} z^* & w^* \\ -w & z \end{pmatrix} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;=\; \epsilon \quad . \tag{9.95}$$

There is a third and last gauge invariant term that one can form using the leptons

$$\ell_L^\dagger\, e_R\, \widetilde{\Phi} \;:=\; \epsilon_{ab}\, \ell_{L\,a}^\dagger\, e_R\, \Phi_b^* \qquad \text{(summation over } a, b = 1, 2 \text{ understood)} \quad . \tag{9.96}$$

The relevant checks are similar to the ones for the analogous quark term (9.90).

The Yukawa interaction term S_{Yukawa} for the standard model is then given by considering the most general linear combination of the three terms (9.86), (9.90) and (9.96) over the three generations of matter fermions. Explicitly, this reads as

$$S_{\text{Yukawa}} \;=\; \sum_{i,j=1}^{3} \int_{\mathbb{R}^4} \left(Y_{ij}\, q_{L\,i}^\dagger\, u_{R\,j}\, \Phi + Y'_{ij}\, q_{L\,i}^\dagger\, d_{R\,j}\, \widetilde{\Phi} + Y''_{ij}\, \ell_{L\,i}^\dagger\, e_{R\,j}\, \widetilde{\Phi} + \text{c.c.} \right) \mathrm{d}x \quad , \tag{9.97}$$

where + c.c. means to add the complex conjugate to produce a real-valued action. This action is by construction invariant under gauge transformations. The three complex 3×3-matrices Y, Y' and Y'' (called *Yukawa couplings*) are free parameters of the standard model that have to be fixed from experiment. These parameters are in particular related to the masses of the quarks and leptons.

Summing up: This completes our description of the standard model action functional

$$S_{\text{SM}} = S_{\text{gauge}} + S_{\text{matter}} + S_{\text{Higgs}} + S_{\text{Yukawa}} \quad . \tag{9.98}$$

The individual terms in this action are displayed in (9.78), (9.82), (9.85) and (9.97). As you may expect from its complexity, the standard model is physically extremely rich. Some of the student projects/presentations will explore these physical aspects further.

The form of the standard model action functional discussed in this chapter is not quite yet ready for perturbative quantization. There are two further aspects that have to be dealt with:

(1) Since the Higgs potential in (9.85) has non-trivial minima, one has to choose a classical vacuum $\Phi_0 \neq 0$ in order to set up a perturbative expansion. In practice, this amounts to writing the Higgs field $\Phi(x) = \Phi_0 + \phi(x)$ in terms of perturbations ϕ of the classical vacuum Φ_0, and then quantize these perturbations with our perturbative QFT techniques. Inserting this decomposition into the standard model action functional, one finds that the terms involving Φ_0 determine mass terms for the matter fermions and the W^+, W^- and Z bosons. The details of this so-called *Higgs mechanism* for spontaneous gauge symmetry breaking will be the topic of some of the student projects/presentations.

(2) To quantize the Yang-Mills fields in (9.78), one requires as in the case of the electromagnetic potential in Chapter 6 a gauge fixing and an analog of the Gupta-Bleuler formalism. For non-Abelian Lie groups, as those present in the standard model, these aspects are more subtle and involved than for the case of electromagnetism. There is a powerful tool, called the *BRST formalism*, that allows one to systematically deal with these issues. A new feature of non-Abelian gauge symmetries is that a consistent gauge fixing procedure requires introducing new fields, the so-called ghost fields, whose role is to cancel the unphysical degrees of freedom in quantum correlation functions and scattering amplitudes. Some of these aspects will be discussed in the student projects/presentations.

Once these two issues have been addressed, one can quantize the standard model of particle physics with our perturbative QFT techniques, which leads to a set of Feynman rules that can be used for making physical predictions for scattering amplitudes. This is how QFT connects to high-energy particle physics.

Further reading

For more details about the standard model of particle physics, see e.g. Srednicki (Chapters 69 and 87–89), Peskin/Schroeder (Chapters 15 and 17) and Schwartz (Chapters 25 and 29) from our reading list in Section 1.3.